Metered Parking Functions

Matt McClinton

UNIVERSITY OF WISCONSIN MILWAUKEE

AMS Central Sectional
Special Session on Parking Functions
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The Team

Figure: Pamela E. Harris (UW Milwaukee), Spencer Daugherty (University of Colorado Boulder), Ian Klein (NC State)

Thanks Kim Harry!

Outline

- What's a Metered Parking Function
  - Not the same as multiple cars parking in the same spot.
  - Time allowed in the parking spot is more important than you think.
- 1-Metered Parking Functions
- Continued Fractions?

Parking Functions

Let $[n] = \{1, 2, \ldots, n\}$.

Definition

Given $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$. We say that α is a **preference vector**, in which cars park under the parking rule,

- Car i parks in its preferred spot a_i, or
- if the preferred spot is taken, it goes to the next available spot, otherwise it leaves the parking lot.

If every car can park, then we say **parking function of length** n and denote the set of parking functions of length n as PF_n

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Alternatively, α is a parking function if and only if $\alpha^{\uparrow} = (a'_1, a'_2, \ldots, a'_n)$ the weakly increasing arrangement of α satisfies for each $i \in [n]$,

$$a'_i \leq i.$$

(m, n)-Parking Functions

What if the number of cars and the number of spots are not the same?

Definition

Let $m, n \in \mathbb{N}$, where m denotes the number of cars and n denotes the number of parking spots. If $\alpha \in [n]^m$ parks all cars under the standard parking rules, we say α is a (m, n)-**parking function**, and we denote the set of (m, n)-Parking functions as $\mathrm{PF}_{m,n}$

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Alternatively, α is an (m, n)-parking function if and only if,

$$|\{k \in [m] : a_k \leq i\}| \geq m - n + i$$

Facts

- The cardinality of the parking functions of length n is,

$$|\mathrm{PF}_n| = \mathrm{pf}_n = (n+1)^{n-1}.$$

- When $1 \leq m \leq n$, the cardinality of the (m, n)-parking functions is

$$|\mathrm{PF}_{m,n}| = \mathrm{pf}_{m,n} = (n-m+1)(n+1)^{m-1}$$

Story Time

"What are some variations of parking functions we can think about?"

\- Pamela

Story Time

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"What if we put meters on the spots and cars have to leave?"
- Kim

Story Time

"What are some variations of parking functions we can think about?"
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"What if we put meters on the spots and cars have to leave?"
- Kim

"That's sick!"
- Matt

Metered Parking Functions.

Now assume that when a car parks, they can only stay parked for a set time allotted.

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Definition (Metered Parking Functions)

Fix a positive integer t, and consider m cars parking in n spots. Cars park under the standard parking rule, except now after car j parks, car $j - t$ (if it exists) will leave as the meter has ran out. If the preference list $\alpha \in [n]^m$ results in all cars parking, we say α is a t-**metered parking function**. We denote the set of t-metered parking functions as $\mathrm{MPF}_{m,n}(t)$. Additionally we denote the cardinality of the sets of metered parking functions as

$$|\mathrm{MPF}_{m,n}(t)| = \mathrm{mpf}_{m,n}(t)$$

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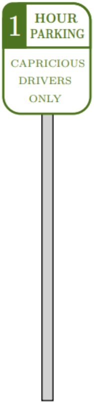
"It takes an hour to find a parking spot"

Example

Consider $\alpha \in (1, 1, 1) \in [2]^3$ with $t = 1$



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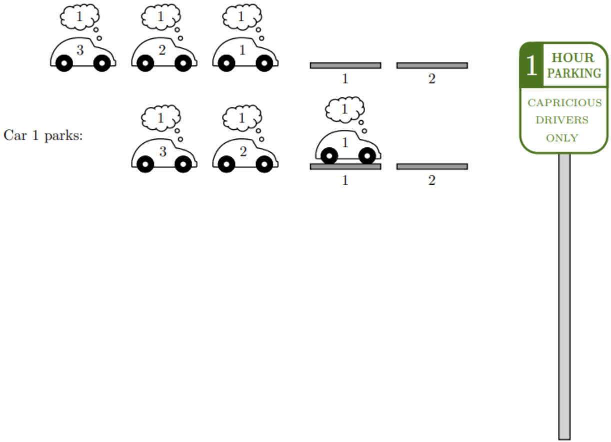

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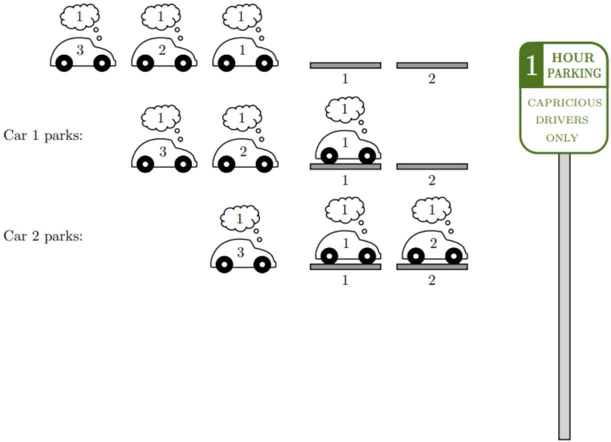

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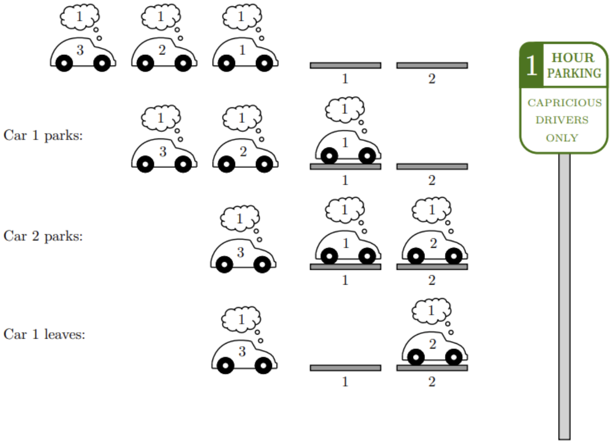

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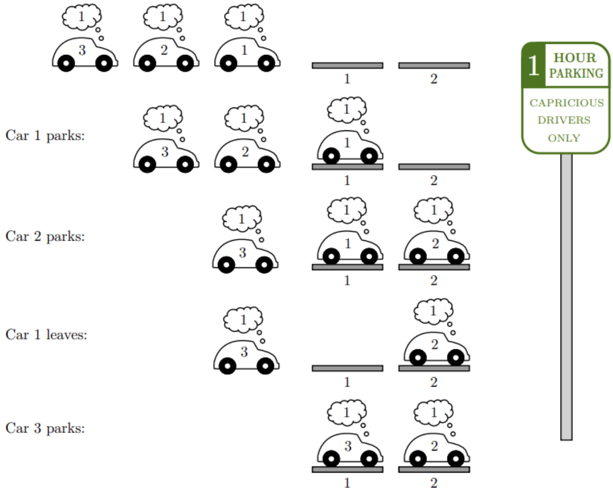

Example

Note: When car 1 parked, that spot became available to car 3. In general for any t, when car i parks, the first car that can park in that spot is car $i + t + 1$.

Pop Quiz!

m cars $\qquad$ n spots $\qquad$ t time

Example

- Let $m, n \in \mathbb{N}$, such that $m \leq n$, and $t \geq m - 1$. Then

$$\mathrm{MPF}_{m,n}(t) =$$

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- Let $m, n \in \mathbb{N}$, such that $m \leq n$, and $t \geq m - 1$. Then

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- Let $m, n \in \mathbb{N}$ be arbitrary, and set $t = 0$. Then

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- Let $m, n \in \mathbb{N}$ such that $m > n$ and $t \geq n$. Then
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Pop Quiz!

m cars n spots t time

Example

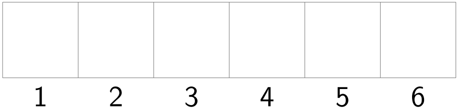
- Let $m, n \in \mathbb{N}$, such that $m \leq n$, and $t \geq m-1$. Then
$$\mathrm{MPF}_{m,n}(t) = \mathrm{PF}_{m,n}.$$
- Let $m, n \in \mathbb{N}$ be arbitrary, and set $t = 0$. Then
$$\mathrm{MPF}_{m,n}(0) = \left[n\right]^{m}.$$
- Let $m, n \in \mathbb{N}$ such that $m > n$ and $t \geq n$. Then
$$\mathrm{MPF}_{m,n}(t) = \emptyset.$$

A Bold Claim

Claim: If $\alpha \in \mathrm{MPF}_{m,n}(t)$, then $\alpha \in \mathrm{MPF}_{m,n}(t')$ for any $t' < t$.

Example

Consider the tuple $\alpha = (3, 3, 3, 3, 4, 5, 6)$ with $t = 2$.



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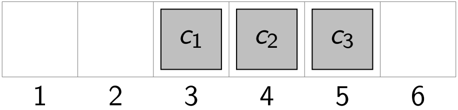

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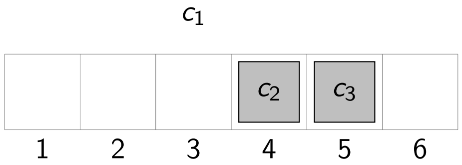

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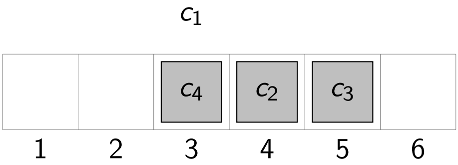

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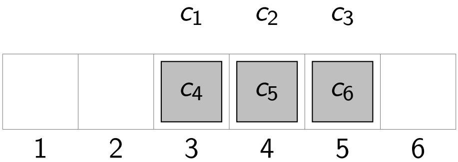

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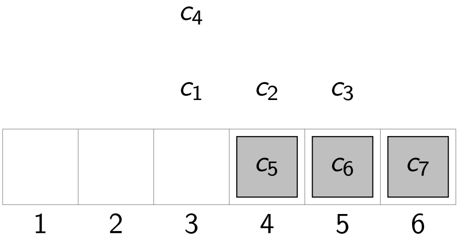

Example

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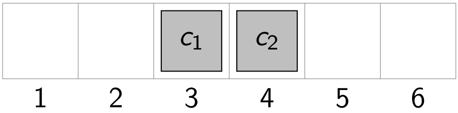

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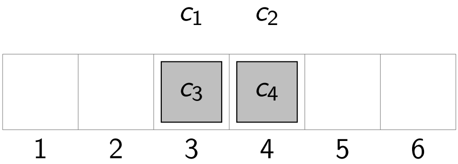

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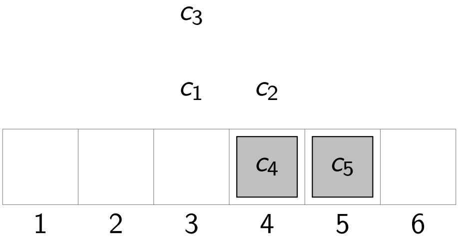

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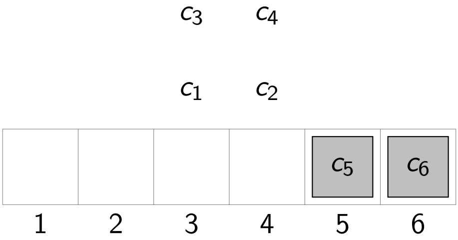

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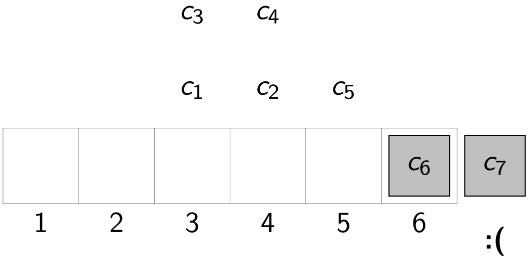
Consider the tuple $\alpha = (3, 3, 3, 3, 4, 5, 6)$ with $t = 1$. **X**



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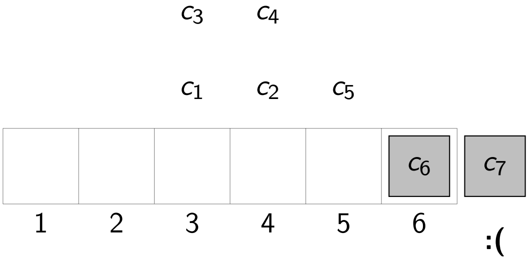

Conjecture: If $t_1 < t_2$, then $\text{mpf}_{m,n}(t_2) \leq \text{mpf}_{m,n}(t_1)$.

$\mathrm{MPF}_{n,n}(1)$

For $m = n$,

Example

$$\mathrm{MPF}_{2,2}(1) = \{(1,1),\ (1,2),\ (2,1)\},$$

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$$\mathrm{MPF}_{2,2}(1) = \{(1,1),\ (1,2),\ (2,1)\},$$

and also

$$2 - \frac{1}{2} = \frac{3}{2}$$

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$$\mathrm{MPF}_{3,3}(1) =$$

$$\left\{\begin{matrix} (1,1,1), & (1,1,2), & (1,1,3), & (1,2,1), & (1,2,2), & (1,2,3), & (1,3,1) \\ (1,3,2), & (2,1,1), & (2,1,2), & (2,1,3), & (2,2,1), & (2,2,2), & (2,3,1) \\ (2,3,2), & (3,1,1), & (3,1,2), & (3,1,3), & (3,2,1), & (3,2,2), & (3,2,3) \end{matrix}\right\}$$

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and also

$$3 - \frac{1}{3 - \frac{1}{3}} = \frac{21}{8}$$

$\mathrm{MPF}_{n,n}(1)$

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Example

$\mathrm{mpf}_{4,4}(1) = 209$

$\mathrm{MPF}_{n,n}(1)$

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Example

$\mathrm{mpf}_{4,4}(1) = 209$ and also

$$4 - \frac{1}{4 - \frac{1}{4 - \frac{1}{4}}} = \frac{209}{56}$$

Your Sequence is...

A097690	Numerators of the continued fraction n-1/(n-1/...) [n times].	+40 11

1, 3, 21, 209, 2640, 40391, 726103, 15003009, 350382231, 9127651499, 262424759520, 8254109243953, 281944946167261, 10393834843080975, 411313439034311505, 17391182043967249409, 782469083251377707328

(list; graph; refs; listen; history; text; internal format)

Continued Fractions?

Theorem (M., Harris, Daugherty, Klein)

For $m \leq n$, the number of $1-$metered parking functions satisfies the recursion

$$mpf_{m+1,n}(1) = n \cdot mpf_{m,n}(1) - mpf_{m-1,n}(1) \tag{1}$$

where $mpf_{1,n}(1) = n$ and we use the convention that $mpf_{0,n} = 1$.

The recursion defined in Equation (1) with $m = n$ corresponds to the OEIS entry `A0097690`, which is the numerator of the continued fraction,

$$n - \cfrac{1}{n - \cfrac{1}{n - \cfrac{1}{n - \dots}}}$$

which terminates after n steps.

Proof of Theorem

$$m \text{ cars} \qquad \leq \qquad n \text{ spots}$$

- Consider the set $\mathrm{MPF}_{m+1,n}(1)$.

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m cars $\leq$ n spots

- Consider the set $\mathrm{MPF}_{m+1,n}(1)$.
- Every $\alpha \in \mathrm{MPF}_{m+1,n}(1)$ is some element $\alpha' \in \mathrm{MPF}_{m,n}(1)$ with an entry appended to the end.

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- How can car c_{m+1} fail to park? It fails to park if car c_m is parked in spot n and car c_{m+1} wants spot n.
- Which means that,

$$
\begin{aligned}
&\mathsf{MPF}_{m+1,n}(1) \\
&= \bigcup_{i=1}^{n} \Big(\{\alpha \in \mathsf{MPF}_{m,n}(1) \text{ with an } i \text{ to appended to the end}\} \\
&\qquad - \{\alpha \in \mathsf{MPF}_{m,n}(1) \text{ where car } m \text{ parks in spot } n\} \Big) \quad (2)
\end{aligned}
$$

Proof of Theorem

m cars $\qquad \leq \qquad$ n spots

- If $i \neq j$, then appending an i to every $\alpha \in \mathrm{MPF}_{m,n}(1)$ is disjoint from the set of $\alpha \in \mathrm{MPF}_{m,n}(1)$ with a j appended to the end.

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- This implies that

$$\begin{aligned}&\mathrm{mpf}_{m+1,n}(1)\\&= n \cdot \mathrm{mpf}_{m,n}(1) - \#\{\alpha \in \mathrm{MPF}_{m,n}(1) : \text{car } m \text{ parks in spot } n.\}\end{aligned}$$

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$$= n \cdot \mathrm{mpf}_{m,n}(1) - \#\{\alpha \in \mathrm{MPF}_{m,n}(1) : \text{car } m \text{ parks in spot } n.\}$$

- $\#\{\alpha \in \mathrm{MPF}_{m,n}(1) : \text{car } m \text{ parks in spot } n\} = \mathrm{mpf}_{m-1,n}(1)$ (see Lemma 3).
- Therefore,

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- $\#\{\alpha \in \mathrm{MPF}_{m,n}(1) : \text{car } m \text{ parks in spot } n\} = \mathrm{mpf}_{m-1,n}(1)$ (see Lemma 3).
- Therefore,

$$\mathrm{mpf}_{m+1,n}(1) = n \cdot \mathrm{mpf}_{m,n}(1) - \mathrm{mpf}_{m-1,n}(1)$$

Summary of Paper

Formulas for the cardinalities of the following sets:

- $\mathrm{MPF}_{n,n}(1)$
- $\mathrm{MPF}_{2,n}(t)$
- $\mathrm{MPF}_{n+k,n}(n-1)$ for $k > 0$.
- $\mathrm{MPF}_{m,2}(1)$
- $\mathrm{MPF}_{m,n}(m-2)$

Some Thoughts.

- What if only some spots have meters?
  - First k spots have a meter? Last k spots?
- What if the meter time isn't the same for every spot?
- Statistics on metered parking functions? Count runs, ascents, descents?
- Check out our open problems!

Figure: Link to the paper

Clown Functions?

From *Parking Functions: Choose your own Adventure* (Harris et. al),

Definition (Clown Functions)

Fix a positive integer d, and consider clowns filling into m cars, where each car can seat d clowns. Clowns enter the cars under the same standard parking rule, except for now we allow d clowns to enter the car. If the preference list $\alpha \in [n]^m$ results in all cars parking, then we say α is a **Clown Function of length** m, and we denote the set of clown functions as $\mathrm{CF}_m(d)$

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Note: This is not the same as metered parking functions!

Example

- $(3, 1, 3) \in \mathrm{MPF}_{3,3}(1)$, and $(3, 1, 3) \notin \mathrm{CF}_3(1)$
- $(1, 3, 3) \notin \mathrm{MPF}_{3,3}(2)$, and $(1, 3, 3) \in \mathrm{CF}_3(2)$