Parking Functions in Higher Dimensions

Catherine Yan

Department of Mathematics
Texas A&M University

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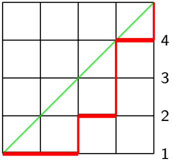
1. Classical Parking Functions

Definition

An integer sequence $\mathbf{a} = (a_1, \ldots, a_n)$ is a parking function of length n iff its non-decreasing rearrangement $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ satisfies $1 \leq a_{(i)} \leq i$ for all $i = 1, \ldots, n$.

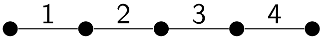
Example: $\boldsymbol{a} = (2, 1, 4, 1)$



$1 \quad 1 \quad 2 \quad 4$

$a_{(1)} \; a_{(2)} \; a_{(3)} \; a_{(4)}$

Vector Parking Functions [Stanley, Kung & Yan]



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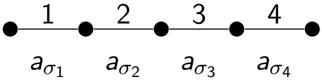

Another way to view Parking Functions $\boldsymbol{a} = (2, 1, 4, 1)$.
Want $a_{\sigma_i} \leq$ weight on the i-th edge for some $\sigma \in \mathfrak{S}_n$.

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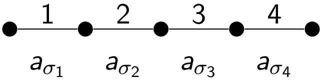

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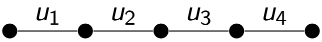

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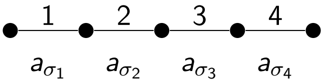

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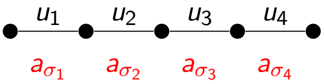
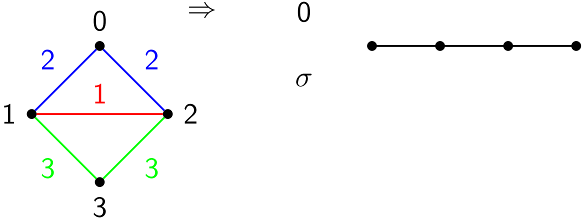


$\boldsymbol{u}$-parking functions are sequences $(a_1, \ldots, a_n) \in \mathbb{Z}^+$ such that $a_{(i)} \leq u_i$.

Graphical Parking Functions [Postnikov & Shapiro]

Let G be an undirected, connected, loopless multigraph with distinguished root 0.
A G-parking function is a sequence $(f_1, \ldots, f_n)$ such that there is a way to rearrange terms as $f_{\sigma_1}, \ldots, f_{\sigma_n}$, and $1 \le f_{\sigma_i} \le$ weight of the i-th edge.

Example: $(5, 2, 3)$.



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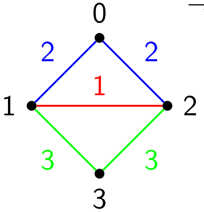
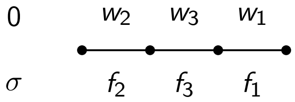


$\Rightarrow$



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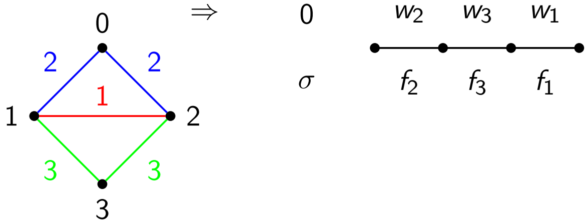

The edge weight on the i-th edge depends on G and σ:

$$w_i = \text{number of edges from } \sigma_i \text{ to } \{0, \sigma_1, \ldots, \sigma_{i-1}\}.$$

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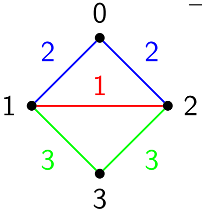

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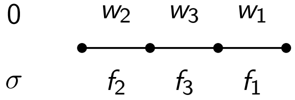

$\sigma = (0)231$

$w_2 = 2 \quad w_3 = 3$

$w_1 = 6$

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Overlap between $\boldsymbol{u}$- and G- Parking functions

Theorem (Gaydarov & Hopkins)

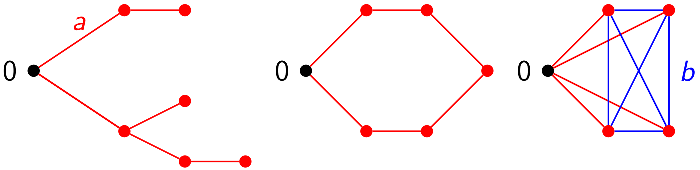
If G is a graph such that $\mathcal{PF}(G)$ is invariant under the action of $\mathfrak{S}_n$, then one of the following cases holds:

(i) $\mathcal{PF}((a, a, \ldots, a)) = \mathcal{PF}(G)$, *where* $a \geq 1$ *and* G *is an* a*-tree;*

(ii) $\mathcal{PF}((a, a, \ldots, a, 2a)) = \mathcal{PF}(G)$, *where* $a \geq 1$ *and* G *is an* a*-cycle;*

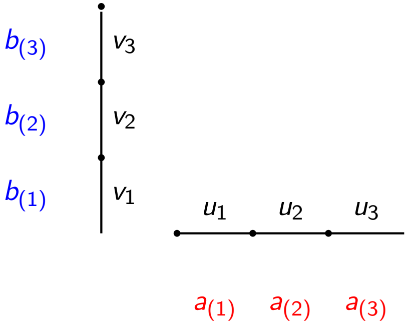
(iii) $\mathcal{PF}((a, a + b, a + 2b, \ldots, a + (n-1)b)) = \mathcal{PF}(G)$, *where* $a, b, n \geq 1$ *and* G *is equal to* $K_{n+1}^{a,b}$.

Otherwise, if $\mathcal{PF}(G)$ is not invariant under the action of $\mathfrak{S}_n$, then there is no $\boldsymbol{u} \in (\mathbb{Z}^+)^n$ such that $\mathcal{PF}(G) = \mathcal{PF}(\boldsymbol{u})$.



Parking functions in 2 dimension: first attempt

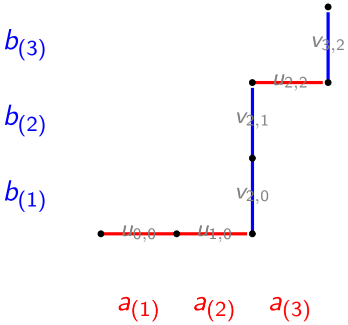
Let $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$



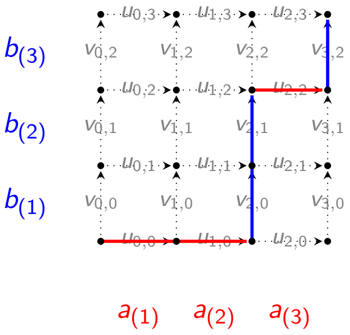
$(\boldsymbol{a}, \boldsymbol{b})$ is just two independent sequences, one in $\mathcal{PF}(\boldsymbol{u})$, the other in $\mathcal{PF}(\boldsymbol{v})$.

Not very interesting!

$\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$.



$\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_q)$. $\boldsymbol{U}$ is the weights on the grid.



2-dimension: try again

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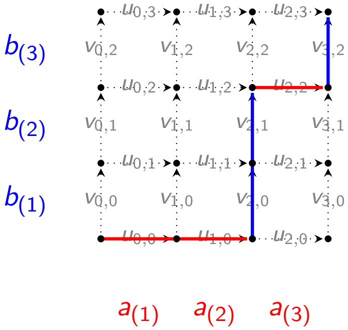
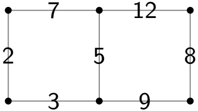


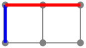
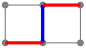
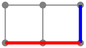
Definition for 2-dim $\boldsymbol{U}$-Parking Functions [Khare, Lorentz, & Y]

$(\boldsymbol{a}, \boldsymbol{b})$ is a 2-dim $\boldsymbol{U}$-parking function if **there exists** a lattice path P from $(0,0)$ to (p,q) whose edge-weights bound the order statistics of $(\boldsymbol{a}, \boldsymbol{b})$.

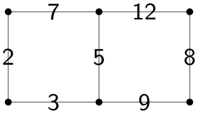
An example



$(\boldsymbol{a}, \boldsymbol{b}) = (a_1, a_2; b_1)$ is a parking function if any of the following happens:

- Path  : $b_1 \leq 2$ and $(a_{(1)}, a_{(2)}) \leq (7, 12)$
- Path  : $b_1 \leq 5$ and $(a_{(1)}, a_{(2)}) \leq (3, 12)$
- Path  : $b_1 \leq 8$ and $(a_{(1)}, a_{(2)}) \leq (3, 9)$

An example



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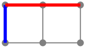
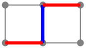
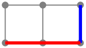
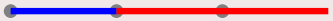
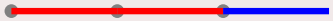
- Path  : $b_1 \leq 2$ and $(a_{(1)}, a_{(2)}) \leq (7, 12)$
- Path  : $b_1 \leq 5$ and $(a_{(1)}, a_{(2)}) \leq (3, 12)$
- Path  : $b_1 \leq 8$ and $(a_{(1)}, a_{(2)}) \leq (3, 9)$

There are $\binom{p+q}{p}$ possible (upper) bounds.





Overlap of G-PF and 2-dimensional $\boldsymbol{U}$-PF: the main case

Joint work with Lauren Snider.

Theorem

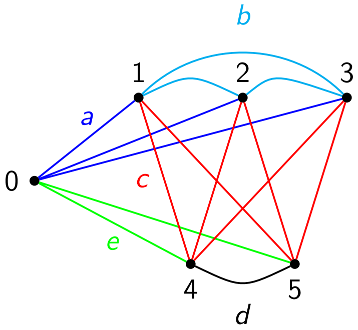
Suppose $\boldsymbol{U} = \{(u_{i,j}, v_{i,j}) : 0 \leq i \leq p, 0 \leq j \leq q\} \subset \mathbb{N}^2$ *is given by*

$$\begin{pmatrix} u_{i,j} \\ v_{i,j} \end{pmatrix} = \begin{pmatrix} b & c \\ c & d \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} a \\ e \end{pmatrix}$$

with $c \in \mathbb{Z}^+$, $a, b, d, e \in \mathbb{N}$, *and at most one of* a, e *is* 0, *then* $\mathcal{PF}^{(2)}_{p,q}(\boldsymbol{U}) = \mathcal{PF}(G)$ *where* $G = K_{p+q+1}$ *with vertex set* $[p+q]_0$ *and edge-weight function*

$$wt_G(\{i,j\}) = \begin{cases} a & \text{if } i = 0 \text{ and } j = 1, \ldots, p; \\ b & \text{if } 1 \leq i < j \leq p; \\ c & \text{if } 1 \leq i \leq p \text{ and } p+1 \leq j \leq p+q; \\ d & \text{if } p+1 \leq i < j \leq p+q; \\ e & \text{if } i = 0 \text{ and } j = p+1, \ldots, p+q. \end{cases}$$

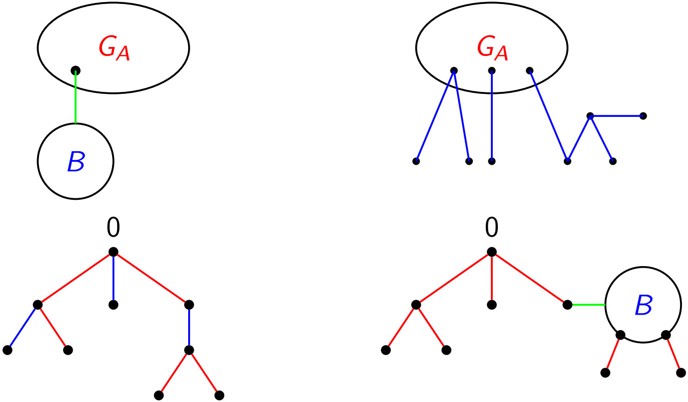
Figure for the main case



When $b = d = 0$, $a = c = e = 1$, this is the case of (p, q)-parking functions introduced by Cori and Poulalhon (2002).

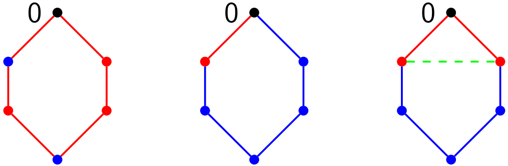
Other cases I: independent $(\boldsymbol{u}, \boldsymbol{v})$-PFs

Merge of G_A, $G_B \in \{$ a-trees, a-cycle, $K^{a,b}_{n+1}\}$, where G_S is the induced graph on $0 \cup S$.

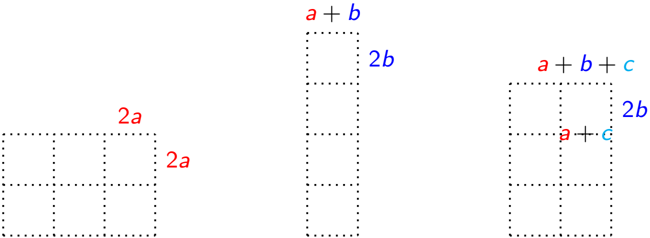


Other case II: special cycles

The graph is cycle-like.



Corresponding weight $\boldsymbol{U}$:



All other weights are a. All other $-$ weights are a and $|$ weights are b.