Lecture 10: Power Series Continued and Graph Coloring
Date: February 23, 2026 **Scribe:** Hunter Pearson

1 Review of Last Lecture

Last time we saw

$$F(x) = \sum_{n \geq 0} f_n x^n$$

where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$

We obtained

$$F(x) = \frac{x}{1 - x - x^2}$$

2 Closed-form Formula for Fibonacci Numbers

If we use partial fractions, we need to factor the denominator of $F(x)$.

Recall:

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus for $1 - x - x^2$, its two roots are

$$r_1 = \frac{-1 + \sqrt{5}}{2} \qquad r_2 = \frac{-1 - \sqrt{5}}{2}$$

Note that $r_1 \cdot r_2 = -1$

Then we get

$$\begin{aligned} 1 - x - x^2 &= -(x - r_1)(x - r_2) \\ &= -\frac{-1}{r_1 \cdot r_2}(x - r_1)(x - r_2) \\ &= \frac{(x - r_1)}{r_1}\frac{(x - r_2)}{r_2} \\ &= (\frac{x}{r_1} - 1)(\frac{x}{r_2} - 1) \\ &= (1 - \frac{x}{r_1})(1 - \frac{x}{r_2}) \end{aligned}$$

This implies

$$F(x) = \frac{x}{(1 - \frac{x}{r_1})(1 - \frac{x}{r_2})} = \frac{A}{1 - \frac{x}{r_1}} + \frac{B}{1 - \frac{x}{r_2}}$$

We need to find A and B. First, we can see that this this is equivalent to

$$x = A(1 - \frac{x}{r_2}) + B(1 - \frac{x}{r_1})$$

Now there are two cases:

Case $x = r_1$:

$$\begin{aligned} r_1 &= A(1 - \frac{r_1}{r_2}) + B(1 - \frac{r_1}{r_1}) \\ &= A(1 - \frac{r_1}{r_2}) + B(0) \\ &= A(1 - \frac{r_1}{r_2}) \\ \implies A &= \frac{r_1}{1 - \frac{r_1}{r_2}} \\ &= \frac{1}{\sqrt{5}} \end{aligned}$$

Case $x = r_2$:

$$\begin{aligned} r_2 &= A(1 - \frac{r_2}{r_2}) + B(1 - \frac{r_2}{r_1}) \\ &= B(1 - \frac{r_2}{r_1} \\ \implies B &= \frac{r_2}{1 - \frac{r_2}{r_1}} \\ &= \frac{-1}{\sqrt{5}} \end{aligned}$$

Together these imply that

$$F(x) = \frac{1}{\sqrt{5}} \left(\sum_{n \geq 0} \frac{1}{r_1^n} x^n \right) - \frac{1}{\sqrt{5}} \left(\sum_{n \geq 0} \frac{1}{r_2^n} x^n \right)$$

One can check that

$$\frac{1}{r_1} = \frac{1}{\frac{-1+\sqrt{5}}{2}} = \frac{2}{-1+\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$$

and

$$\frac{1}{r_2} = \cdots = \frac{1-\sqrt{5}}{2}$$

This gives us

$$F(x) = \frac{1}{\sqrt{5}} \left(\sum_{n \geq 0} \left(\frac{1+\sqrt{5}}{2} \right)^n x^n \right) - \frac{1}{\sqrt{5}} \left(\sum_{n \geq 0} \left(\frac{1-\sqrt{5}}{2} \right)^n x^n \right)$$

Recall that $F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots$, which implies

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

This is an integer!

3 Catalan Generating Function

Recall $c_0 = 1$ and $c_n = \sum_{j=1}^{n} c_{j-1}c_{n-j}$ for $n \geq 1$.

Step 1

The above implies that

$$c_n x^n = \left(\sum_{j=1}^{n} c_{j-1}c_{n-j} \right) x^n$$

Which implies

$$\sum_{n\geq 1} c_n x^n = \sum_{n\geq 1} \left(\sum_{j=1}^{n} c_{j-1}c_{n-j} \right) x^n$$

Step 2

Let $C(x) = \sum\limits_{n\geq 0} c_n x^n$

The left hand side of the equation from step one gives us

$$\sum_{n\geq 1} c_n x^n = C(x) - c_1 = C(x) - 1$$

The right hand side of the equation from step one gives us

$$\begin{aligned}
\sum_{n\geq 1} \left(\sum_{j=1}^{n} c_{j-1}c_{n-j} \right) x^n &= x \left(\sum_{n\geq 1} \left(\sum_{j=1}^{n} c_{j-1}c_{n-j} \right) x^{n-1} \right) \\
&= x \left(\sum_{n\geq 1} \left(\sum_{j=0}^{n-1} c_j c_{n-j-1} \right) x^{n-1} \right) \\
&= x \left(\sum_{n\geq 0} \left(\sum_{j=0}^{n} c_j c_{n-j} \right) x^n \right) \\
&= xC(x)C(x)
\end{aligned}$$

Step 3

So $C(x) - 1 = xC(x)^2$.

Solving for $C(x)$ using the quadratic formula gives us $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$

There are two issues:

- There seems to be two generating functions, and this cannot be.
- $\frac{1}{x}$ does not make sense as a formal power series, as x has no inverse.

This raises a question: Which sign should we use, $+$ or $-$?

Idea: Choose the sign such that the whole numerator $1 \pm \sqrt{1-4x}$ has no constant term. In other words, we want $1 \pm \sqrt{1-4x} = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + \dots$ where $w_0 = 0$.

If we can do this, then we shift the sequence one step to the left, i.e. $\frac{1\pm\sqrt{1-4x}}{x} = w_1 + w_2x + w_3x^2 + w_4x^3 + \dots$. This will take care of both problems.

There is a generalization of the binomial theorem where for any $n \in \mathbb{Q}$, we have

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k, \tag{1}$$

where the binomial coefficient is again generalized to work when n is rational. We will not cover this generalization, but note that $\binom{n}{0} = 1$, even if $n = \frac{1}{2}$, $k = 0$.

Since $\sqrt{1-4x} = (1-4x)^{\frac{1}{2}}$, we can use this generalization to find that $(1-4x)^{\frac{1}{2}} = 1 + w_1x + w_2x^2 + \dots$. Then $1 - (1-4x)^{\frac{1}{2}} = 0 - w_1x - w_2x^2 - w_3x^3 - \dots$, which is what we needed.

This implies that we should choose the minus sign in order to solve our issues.

Thus $C(x) = \frac{1-\sqrt{1-4x}}{2x}$

Step 4

One can recover $c_n = \frac{1}{n+1}\binom{2n}{n}$ by using $C(x)$ and equation (1)

4 Graph Coloring and Chromatic Polynomials

Let $G = (V, E)$ be a graph.

A vertex coloring of G is a mapping $c : V \to S$, where S is a finite set of colors.

For example, $S = \{\text{red, black, blue}\}$. Sometimes it is easier to label these colors as numbers instead. i.e let $S = \{1, 2, 3\}$.

A coloring is proper if for all $e = \{u, v\} \in E$, it holds that $c(u) \neq c(v)$.

Some Examples

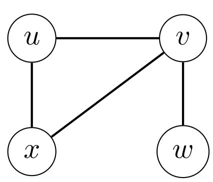

Figure 1: Original Graph

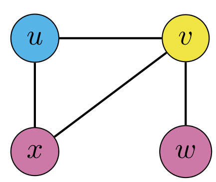

Figure 2: A Proper Coloring

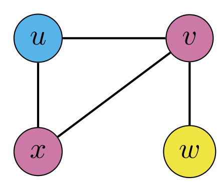

Figure 3: Not A Proper Coloring

Chromatic Number

The chromatic number of G, denoted $\chi(G)$ is the minimum cardinality of the set of colors S such that a proper coloring exists.

i.e. in $c : V \to S$, what is the minimum $|S|$ such that a proper coloring is possible.

From Our Example

1. $\chi(G) \leq 3$ as we produced a proper coloring with $|S| = 3$ different colors.

2. Also, we have the 3 cycle $u - v - x - u$. Any 3 cycle needs at least 3 colors. Thus $\chi(G) \geq 3$

Thus since $3 \leq \chi(G) \leq 3$, we have $\chi(G) = 3$.
Note: In general, for any $k \in \mathbb{N}$, $\chi(K_n) = n$, where K_n is the complete graph on n nodes.

Counting Proper Colorings

How many proper colorings of G with t colors are there?
We write $P(G, t)$ to denote the number of proper colorings on V using t colors.
$P(G.t) = |\{c : V \to [t]| \ c \text{ is proper}\}|$
Note: By definition of $\chi(G)$, $P(G, t) = 0$ for $0 \leq t < \chi(G)$ and $P(G, t) > 0$ for $t \geq \chi(G)$

From Our Example

- u has t options
- v has $t - 1$ options (color must be different than the color of u)
- x has $t - 2$ options (color must be different than the color of v and u, which are themselves different)
- w has $t - 1$ options (color must be different than color of v)

So $P(G, t) = t(t - 1)(t - 2)(t - 1) = t^4 - 4t^3 + 5t^2 - 2t$

4.1 Next Time

- We will show $P(G, t)$ is indeed a polynomial no matter what G is.
- We will see what graph theoretic information about G is embedded in $P(G, t)$.