

Lecture 3: The Principle of Inclusion-Exclusion (PIE) and Applications
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Recall the sum principle:

$$|S \sqcup T| = |S| + |T|$$

The disjoint union is required.

You may recall from prior life that 1)

$$|S \cup T| = |S| + |T| - |S \cap T|$$

Every element in the intersection is counted twice in the first part. The double-counting must be corrected.

Goal: Generalize 1) to work with any finite collection

$$S_1, S_2, \dots, S_n \subseteq S$$

For $n=2$,

$$|S| = |S_1 \cup S_2| + |S \setminus (S_1 \cup S_2)|$$

So if we know $|S|$ and $|S \setminus (S_1 \cup S_2)|$, then we know $|S_1 \cup S_2|$. We will focus on terms like $|S \setminus (S_1 \cup S_2)|$.

Theorem 0.1. *The Principle of Inclusion and Exclusion: Let S be finite and $S_1, S_2, \dots, S_n \subseteq S$. Then,*

$$|S \setminus \bigcup_{1 \leq i \leq n} S_i| = |S| - \sum_{1 \leq i \leq n} |S_i| + \sum_{1 \leq i < j \leq n} |S_i \cap S_j| - \dots + (-1)^n \left| \bigcap_{1 \leq i \leq n} S_i \right|$$

Proof. It suffices to show that for any s in S , its contribution to LHS is the same as its contribution to RHS.

Consider any s in S ; there are two cases:

1)

$$s \notin \bigcup_{i=1}^n S_i$$

LHS: So, $s \in \bigcup_{i=1}^n S_i$ which contributes 1 to the count.

RHS: s is in S , so it is accounted for in $|S|$. s is not in any of the intersections, so in total it contributes 1 to the count of RHS.

2)

$$s \in \bigcup_{i=1}^n S_i$$

. Suppose $s \in S_i$ for exactly $m \geq 1$ of the sets S_1, S_2, \dots, S_n

LHS:

$$s \notin S \setminus \bigcup_{i=1}^n S_i$$

so it contributes 0 to LHS.

RHS: We know that $s \in S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_m}$, a k -subset of the m subsets that contain s . Based on the alternating signs,

$$0 = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \dots + (-1)^m \binom{m}{m}$$

because

$$\sum_{k \text{ is odd}} \binom{m}{k} = \sum_{k \text{ is even}} \binom{m}{k}$$

□

Application 1: Derangements

A permutation is a bijection

$$\pi : [n] \rightarrow [n]$$

For example, in one line notation and function notation we have

$$\pi = 4, 1, 2, 3$$

$$\pi(1) = 2, \pi(2) = 3, \pi(3) = 4, \pi(4) = 1$$

A permutation is a derangement if $\pi(i) \neq i \forall i \in [n]$.

How many derangements of $[n]$ are there?

Idea: use PIE to “count with restrictions”:

1. S : the set of all objects
2. S_i : the subset of S that violates the i th restriction
3. $S \setminus \bigcup_{i=1}^n S_i$: the subset of S that meet all restrictions.

In the case of derangements,

1. $S = \mathcal{S}_n$, the set of all permutations on $[n]$
2. $S_i \subseteq \mathcal{S}_n$ such that $\pi \in S_i \iff \pi(i) = i$
3. $S \setminus \bigcup_{i=1}^n S_i$ is the set of derangements

So the number of derangements is $D(n) = |S \setminus \bigcup_{i=1}^n S_i|$

$\pi = 1, 2, \dots, n \implies \pi \in S_i \forall i \in [n]$. The intersections over S_1, \dots, S_n are non-trivial.

Note that any $\pi \in S_i$ looks like

$$\pi^{-1}(1), \pi^{-1}(2), \dots, \pi^{-1}(i-1), i, \pi^{-1}(i+1), \dots, \pi^{-1}(n)$$

Except for the i , π is a permutation on $n-1$ letters.

$$|S_i| = (n-1)!$$

There are $\binom{n}{1}$ choices for S_i , so

$$\sum_{i=1}^n |S_i| = \binom{n}{1} (n-1)! = n(n-1)! = n!$$

Similarly, any $\pi \in S_i \cap S_j$ looks like

$$\pi^{-1}(1), \dots, \pi^{-1}(i-1), \pi^{-1}, i, (i+1), \dots, \pi^{-1}(j-1), j, \pi^{-1}(j+1), \dots, \pi^{-1}(n)$$

Except for i and j , π is a permutation on $n-2$ letters.

$$|S_i \cap S_j| = (n-2)!$$

There are $\binom{n}{2}$ options for S_i, S_j , so

$$\sum_{i < j} |S_i \cap S_j| = \binom{n}{2} (n-2)!$$

In general, for any k number of fixed points, the sum of k -term intersections:

$$\binom{n}{k} (n-k)! = \frac{n!}{k!}$$

To sum up (pun intended),

$$D(n) = |S \setminus \bigcup_{i=1}^n S_i| = n! - n! + \frac{n!}{2} - \frac{n!}{3} + \dots + (-1)^n \frac{n!}{n!}$$

$$D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right)$$

For any $n \geq 0$.

Corollary: $\lim_{n \rightarrow \infty} \frac{D(n)}{n!} = \frac{1}{e}$

Proof. For any fixed point n , $\frac{D(n)}{n!}$ is a truncation of the Taylor series approximation for $\frac{1}{e}$. □

So a random permutation has a 37 percent chance of being a derangement.

Application 2: Chromatic Polynomials

Given $G = (V, E)$

1. $S = \{c : V \rightarrow [t]\}$
2. For any $e = \{u, v\}$, $S_e \subseteq S$ such that $c(u) = c(v)$
3. $S \setminus \bigcup_{e \in E} S_e$ is the set of proper colorings.

We want $|S \setminus \bigcup_{e \in E} S_e|$

$$|S| = t^n \text{ for } n = |V|$$

Consider any $F \subseteq E$ and the induced subgraph $H_F = (V, F)$.

Note that any $c \in \bigcap_{e \in F} S_e$ makes all edges in H_F monochromatic, so each connected component of H_F is monochromatic.

$\kappa(H_F)$ is the number of connected components of H_F .

$|\bigcap_{e \in F} S_e| = t^{\kappa(H_F)}$ since the coloring of the different components are independent of one another.

Using PIE, we have that:

$$|S \setminus \bigcup_{e \in E} S_e| = \sum_{\emptyset \subseteq F \subseteq E} (-1)^{|F|} t^{\kappa(H_F)} = P(G; t)$$

Some observations:

1. if $|F| = 0$, $\kappa(H_F) = n$.
2. if $|F| = 1$, we are picking individual edges, so $\kappa(H_F) = n - 1$ and the coefficient in front of t^{n-1} is the number of edges $|E|$.
3. if $|F| = |E|$, then $\kappa(H_F) = \kappa(G)$, so the smallest power of t with a nonzero coefficient is $t^{\kappa(G)}$.
4. $P(G;t)$ is a polynomial.