Lecture 12**:** Graph Coloring (continued) and Matchings
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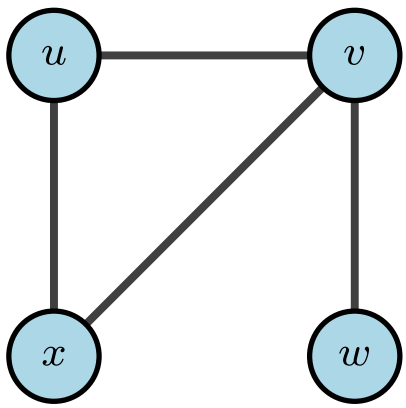
1 Colorings

Some additional properties of graph coloring.

- Given $G = (V, E)$
- Let $G\backslash e$ be the graph obtained by deleting E.
- Let $G = G/e$ be the graph obtained by contracting e.



Lemma:

$$P(G;t) = P(G\backslash e;t) - P(G/e;t)\ \forall e \in E$$

Proof

$$P(G;t) + P(G/e;t) = P(G\backslash e;t)$$

Consider any $e = \{u, v\} \in E$ Consider the proper colorings of $G\backslash e$. There are two options:

- $C(u) \neq C(v) \implies$ induces a proper coloring of G
- $C(u) = C(v) \implies$ induces a proper coloring of G/e

□

Note

- The lemma gives us a way of building $P(G;t)$ recursively
- The coefficients of $P(G;t)$ have a combinatorial interpretation in terms of the number of $F \subseteq E$ of a given size with some additional (hard to explain here) properties (***Final Project Idea***)

2 Matchings

Let $G = (V, E)$
A **matching** of G is a set $M \subseteq E$ that is pairwise non-incident.
A **perfect matching** is a matching M that covers all of V
i.e. $\bigcup_{e \in E} e = V$

The matching number of G, call it $\alpha(G)$, is the maximum size of a matching in G.

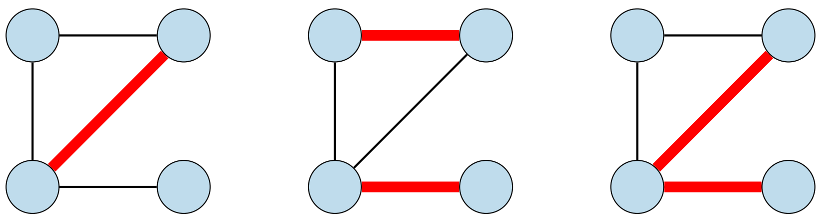


Figure 1: A matching, perfect matching, and not a matching.

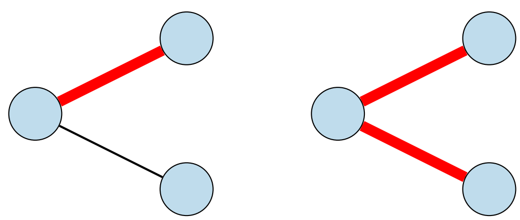


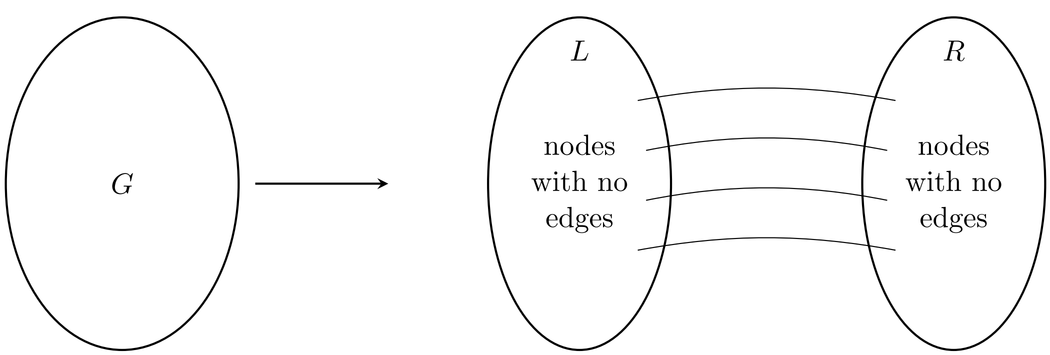
Figure 2: A matching and not a matching. Since $\alpha(G) = 1 < 3/2$, this graph has no perfect matching.

So when do perfect matchings exist? How do we find large matchings?

2.1 Bipartite Graphs

In a graph $G = (V, E)$, $S \subseteq V$ is an independent set if its nodes are pairwise non-adjacent
i.e. $\forall u, v \in S,\ \{u, v\} \notin E$

A graph is **Bipartite** if you can write $V = L \sqcup R$ where $L,\ R$ are independent sets.



For $u \in V$, let $N(u) = \{v \in V : \{u, v\} \in E\}$

For $S \subseteq V$, let $N(S) = \bigcup_{u \in S} N(u)$
In an X, Y-Bipartite Graph (say X is left and Y is right), if a matching of size $|X|$ exists, then

$$(*)\ |N(S)| \geq |S|,\ \forall S \subseteq X \text{ (Hall's Condition)}$$

As it turns out, (*) is also sufficient.

Hall's Theorem: Let $G = (X \sqcup Y,\ E)$ be a bipartite graph. There exists a matching of size $|X|$ if and only if $|N(S)| \geq |S|,\ \forall S \subseteq X$

Proof
$\Longrightarrow$ If such a matching exists one can readily check that the condition (*) holds.

$\Longleftarrow$ We prove by induction on $|X|$.

If $|X| = 1$, the claim follows.

Inductive Step

Assume the statement is true for all graphs whose left side is the size $|X| - 1$

Case 1 $|N(S)| > |S|,\ \forall \emptyset \neq S \subsetneq X$

- consider any $x \in X$ and $y \in N(x)$. Then such a y exists since (*) holds when $S = \{x\}$
- Let $G' = G - \{x, y\}$
  G' satisfies Hall's condition $\forall S \subseteq X - \{x\}$, since each such S loses at most one neighbor, y.
- By assumption, $|N_G(S)| \geq |S| \implies$ by inductive hypothesis $\exists$ a matching M' of G' of size $|X| - 1$. Then $M = M' + \{x, y\}$ is a matching G of size $|X|$.

Case 2 $\exists \emptyset \neq S \subsetneq X$ such that $|N_G(S)| = |S|$

Pick any such S. Let $G_1 = (S \sqcup N(S), E(S \sqcup N(S)))$ and let $G_2 = G - G_1$

WTS that (*) holds for G_1 and G_2

(*) holds for G_1 and all $\emptyset \neq T \subsetneq S$ because $N(T) \subseteq N(S)$ and all edges between S and $N(S)$ are preserved in G_1 (and so all edges between T and $N(T)$ are preserved in G) and we had $|N(T)| \geq |T|$ since (*) holds in G.

(*) holds for G_2 and all $\emptyset \neq T \subseteq X - S$

Note that $N_{G_2}(T) = N_G(T \cup S) \backslash N_G(S)$

$$\begin{aligned} |N_{G_2}(T)| &= |N_G(T \cup S) \backslash N_G(S)| \\ &= |N_G(T \cup S)| - |N_G(S)| \\ &\geq |S \cup T| - |N_G(S)| \\ &= |S \cup T| - |S| \\ &= |T| \end{aligned}$$

$$\implies |N_{G_2}(T)| \geq |T|$$

By induction G_1 has a matching M_1 of size $|S|$ and G_2 has a matching M_2 of size $|X - S|$
$M = M_1 + M_2$ is a matching of G of size $|X|$

□

Observations

- Hall's theorem $\implies$ fast verification of "yes" or "no" answer to the question "does there exist a matching that covers X"?
  - "yes" is verified with an appropriate M.
  - "no" is verified with appropriate $S \subseteq X$
- Hall's theorem does not tell you how to decide between "yes" and "no" in a timely manner (we will see how to decide fast later).

Corallary (Marriage Theorem): For any $k \geq 1$, if G is a $k-$regular bipartite graph, then it contains a perfect matching

- $k-$regular means $deg(u) = k, \ \forall u \in V$.

Proof

Consider any $S \subseteq X$, then by the regularity property/assumption,

$$k|S| \leq k|N(S)|$$

Left hand side is the number of edges into $N(S)$ coming from S and RHS is the number of edges into $N(S)$ since G is $k-$regular.

Since $k \geq 1$ we can divide by k on both sides,

$$|S| \leq |N(S)|$$

□

Project Idea: Stable marriage theorem which accounts for ordered preferences.