Lecture 14: Matchings Continued

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Brief Detour Into Carlos' View of the world

Acronym: A.C.O: Algorithms, Combinatorial, and Optimization. (The Barriers between these are blurred. Algorithms: Combinatorial algorithms like the augmenting path algorithm we saw last time. Common from a computer science lens. Combinatorics: graph theoretic properties, like Halls theorem, common from math lens. Optimization: "polyhedral approach" uses linear algebra and "math programming" for modeling and optimization. Common from operations research lens. Polyhedral approach: Idea is to associate to a set $\mathcal{C}$ of combinatorial objects you care about to points in space and use geometry of these points to find the "best".

For example, consider matchings of a graph $G = (V, E)$. For each edge $e \in E$, associate to it a binary decision variable $x_e \in \{0, 1\}$. Then finding a maximum in G is the same as the following linear program:

$$\begin{aligned} \text{Maximize: } & \sum_{e \in E} x_e \\ \text{Such That: } & \sum_{e \in \delta(\mu)} x_e \leq 1 \forall \mu \in V \\ & x_e \in \{0, 1\} \forall e \in E \end{aligned}$$

So a matching M is a point $X_M \in \{0, 1\}^E$. $X_{M_e} = 1 \iff e \in M$.

- Note: There are specialized computer software to solve problems like IP
- In this sense, it give you lots of modeling flexibility
- In general, problems with 0,1 variables are "Hard"
- If G happens to be bipartite, you can replace $x_e \in \{0, 1\}$ for $0 \leq x_e \leq 1$ to obtain a linear programming relaxation that happens to produce integer solutions. It is really special when this happens because LP's are "easy".
- If G is not bipartite, the story is more complicate but still an exact LP formulation is possible.
- Combinatorial optimization/mathematical programming study how to leverage LP's/IP's to get "good" solutions

What if Halls Condition Doesn't Hold?

Recall Halls Condition:

$$|N(S)| \geq |S|, \forall S \subseteq X \tag{$\star$}$$

Halls Theorem tells us that if $(\star)$ holds for all $S \subseteq X$, then there exists a matching of size $|X|$. Question: What if the condition doesn't hold for some $S \subseteq X$? How large of a matching exists? For $\emptyset \neq S \subseteq X$, let the defect of S be

$$\text{df}(S) := |S| - |N(S)|$$

.

Halls condition becomes $\text{df}(S) \leq 0 \forall S \subseteq X$.

To answer our question, it is natural to think about the worst case, i.e. the largest defect.

Define

$$d^* = \max_{\emptyset \neq S \subseteq X} \text{df}(S)$$

Also, let $\alpha(G)$ be the size of the largest matching in $G = (X \sqcup Y, E)$.

Corollary 1.1. *(Ore's Defect Formula) Let $G = (X \sqcup Y, E)$ be bipartite. Then $\alpha(G) = |X| - d^*$*

Proof. Every matching misses to match $\text{df}(S)$ nodes from $S \subseteq X$. This implies

$$|\alpha(G)| \leq |X| - \text{df}(S) \forall S \subseteq X$$

Then

$$\begin{aligned}\alpha(G) &\leq \min_{\emptyset \neq S \subseteq X} \{|X| - \text{df}(S)\} \\ &= |X| - \max_{S} \text{df}(S) \\ &= |x| - d^*\end{aligned}$$

Thus we have

$$\alpha(G) \leq |x| - d^*$$

Now we want to show that $\alpha(G) \geq |x| - d^*$. It suffices to produce a matching of G of size $|x| - d^*$.

Idea: Produce a new graph G' by adding d^* dummy nodes to Y, each of which is connected to all of x.

Claim: G' meets Halls Condition $\forall S \subseteq X$.

This holds since for each $S \subseteq X$, $N(S)$ increased by d^* and $\text{df}(s) \leq d^*$.

The new defect of S is less than or equal to 0.

Then G' has a matching M of size $|X|$ (by Hall's Theorem)

Now $M = M_1 \sqcup M_2$, where M_2 is the subset of edges of M that connect to dummy nodes, and M_1 is the rest. Then $|M_2| \leq d^*$ since there are only d^* dummy nodes.

Finally, M_1 is a matching in G (the original graph)

Since it does not use dummy nodes, we know

$$|M_1| = |M| - |M_2| = |X| - |M_2| \geq |X| - d^*$$

□

The Minimum Vertex Cover Problem

A new problem: the minimum vertex cover problem.

A vertex cover (i.e. a cover made up of vertices) of a graph $G = (V, E)$ is a subset of the nodes such that every edge has at least one endpoint in S.

Thus $e \cap S \neq \emptyset \forall e \in E$.

The goal is to find a vertex cover that is as small as possible. Let $\beta(G)$ be the minimum size of a vertex cover in G.

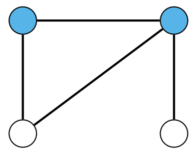

Figure 1: A Vertex Cover

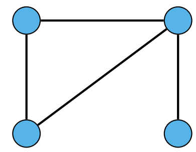

Figure 2: Another Vertex Cover

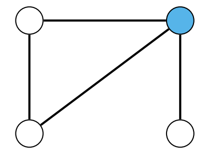

Figure 3: Not A Vertex Cover

Theorem 1.2. *(König-Egerváry) If G is a bipartite graph, then $\alpha(G) = \beta(G)$*

Proof. $\alpha(G) \leq \beta(G)$: Let M^* be a maximum matching. No node in a vertex cover can cover more than 1 edge in M^* because the edges in M^* are non adjacent. So you need at least $m^* = \alpha(G)$ nodes to cover G

$\alpha(G) \geq \beta(G)$: It suffices to produce a vertex cover $S \subseteq X$ such that $|S| = \alpha(G)$. Since by definition of $\beta(G)$, we will have $\alpha(G) = |S| \geq \beta(G)$. By Ore's defect formula, we have that

$$\alpha(G) = |x| - \mathrm{df}(T) = |X| - |T| + |N(T)|$$

for some $T \subseteq X$ with $\mathrm{df}(T) = d^*$. Note that $N(T) \sqcup (X - T)$ is a vertex cover.

Finally, the size of $N(T) \sqcup (X - T)$ is $|N(T)| + |X| - |T| = \alpha(G)$. □

This theorem gives you a min-max relation. Thus it allows us a certificate of optimality.

For bipartite graphs, say M^* is a matching and I claim it is a maximum matching. I can convince you of this by producing a vertex cover S of size $|S| = |M^*|$, and vice versa. This kind of relation is in some ways akin to LP duality.