Lecture 15: Connectivity

Date: March 11, 2026 **Scribe:** Parsa S. Farahani

1 Motivation

We have looked at connected graphs, but some graphs are better connected than others. For example, trees are very fragile because removing any one node/edge breaks connectivity. On the other hand, a complete graph is very well connected.

Formalize "how well connected a graph is":

Definition 1.1. *A **vertex cut** (also known as a separating set) of a connected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $G - S$ has more than one component.*

Definition 1.2. *We say G is k-vertex-connected if:*

   i) $|V| > k$.

   ii) Every vertex cut of G has size at least k.

Definition 1.3. *The **vertex connectivity** of G is:*

$$\kappa(G) = \max\{k \in \mathbb{N} : G \text{ is } k\text{-vertex-connected}\}$$

i.e., $\kappa(G)$ is the smallest size of a vertex cut (assuming one exists).

For example:

- If G is a tree, then $\kappa(G) = 1$.

- If G is a cycle (with $|V| \geq 3$), then $\kappa(G) = 2$.

- If $G = K_n$ is the complete graph on n nodes, then $\nexists$ vertex cuts so the second requirement of k-vertex-connectivity holds vacuously. But by requirement i) in which $|V| > k$, we have that $\kappa(K_n) = n - 1$.

2   Edge Connectivity

Definition 2.1. *A **disconnecting set** of edges of a connected multigraph $G = (V, E)$ is a subset $F \subseteq E$ such that $G - F$ is disconnected.*

We say G is k**-edge-connected** if every disconnecting set has size at least k.

Definition 2.2. *The **edge connectivity** of G is:*

$$\kappa'(G) = \max\{k \in \mathbb{N} : G \text{ is } k\text{-edge-connected}\}$$

i.e., $\kappa'(G)$ is the smallest size of a disconnecting set.

For $\emptyset \neq S \subsetneq V$, let $\delta(S) \subseteq E$ be the subset of edges with one endpoint in S and the other in $V \setminus S$. We say $\delta(S)$ is an **edge cut**. Note that $\delta(S)$ is a disconnecting set.

Definition 2.3. *A **minimal** set of edges whose deletion increases the number of components is a **bond**.*

Proposition 2.4. *For any graph $G = (V, E)$, any minimal disconnecting set is an edge cut.*

Proof. Suppose $G - F$ is disconnected. Let $T \subseteq V$ be the nodes of one of the components of $G - F$. Then $\delta(T) \subseteq F$. Since $\delta(T)$ is a disconnecting set, F is not minimal unless $\delta(T) = F$. $\qquad \square$

Proposition 2.5. *If G is connected, then $\delta(S)$ is a **bond** (minimal disconnecting set) if and only if the induced subgraphs $G[S]$ and $G[V \setminus S]$ are themselves connected.*

Proof. ($\Rightarrow$) If G, $G[S]$, and $G[V \setminus S]$ are themselves connected, then no proper subset of $\delta(S)$ disconnects S and $V \setminus S$, so $\delta(S)$ is a bond.
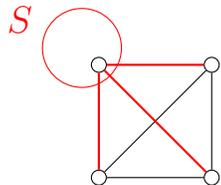
($\Leftarrow$) Conversely, suppose without loss of generality that $G[S]$ is not connected. Let $T \subsetneq S$ be the nodes in one of its components. Then, $\delta(T) \subsetneq \delta(S)$ since

$$\delta(T, S \setminus T) = \{e = \{u, v\} \in E : u \in T, v \in S \setminus T\} = \emptyset.$$
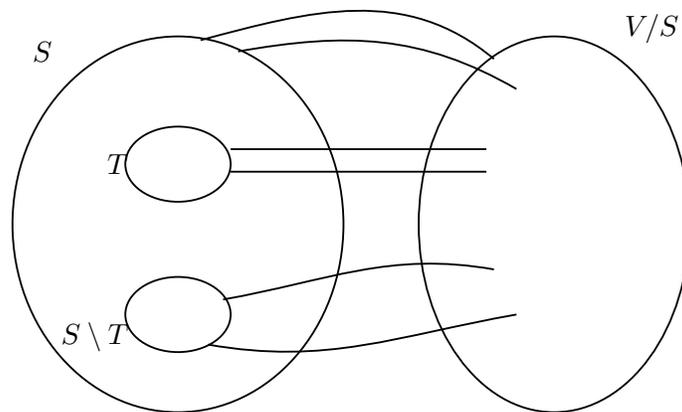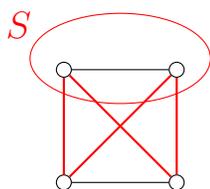

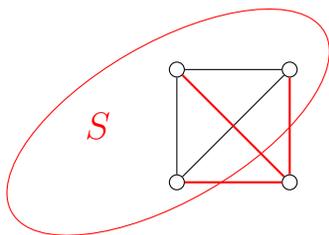$\delta(T)$ is a disconnecting set, so $\delta(S)$ is not a bond. $\square$

Example 2.6. *Consider K_n. Every edge cut in K_n has size $k(n-k)$ for some $1 \leq k \leq n-1$. For $n = 4$:*

$$|\delta(S)| = 3 = 1.(4-1).$$

$$|\delta(S)| = 4 = 2.(4-2)$$

$$|\delta(S)| = 3 = 3(4-1)$$

3   Whitney's Theorem

Let $\deg(G) = \min_{u \in V}\{\deg(u)\}$.

Theorem 3.1 (Whitney). *For graph $G = (V, E)$, we have that*

$$\kappa(G) \leq \kappa'(G) \leq \deg(G).$$

Proof. Note that edges in $\delta(u)$ for any $u \in V$ is a disconnecting set.

$$\kappa'(G) \leq |\delta(u)|, \quad \forall u \in V$$

$$\Rightarrow \kappa'(G) \leq \min_{u \in V}\{|\delta(u)|\} = \deg(G).$$

To prove that $\kappa(G) \leq \kappa'(G)$, let

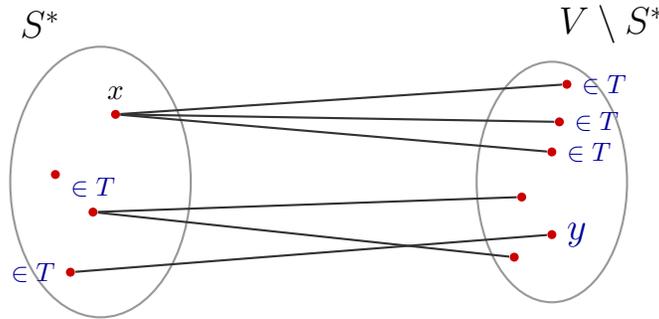$$S^* \in \operatorname{argmin}_{\emptyset \neq S \subsetneq V}|\delta(S)|.$$

First, suppose that all of S^* is adjacent to all of $V \setminus S^*$. In this case,

$$\underbrace{\kappa'(G) = |\delta(S^*)|}_{\substack{\uparrow \\ \text{by def of } \kappa'(G) \\ \text{and first proposition of} \\ \text{today}}} = \underbrace{|S^*| \cdot |V \setminus S^*| \geq |V| - 1 \geq}_{\substack{\uparrow \\ \text{arithmetic fact}}} \underbrace{\kappa(G)}_{\substack{\uparrow \\ \text{requirement of } k\text{-vertex} \\ \text{connectivity requiring } |V| > k}}$$

This proves $\kappa'(G) \geq \kappa(G)$ under our assumption.

Otherwise, $\exists x \in S^*$ and $y \in V \setminus S^*$ s.t. $\{x, y\} \notin E$.

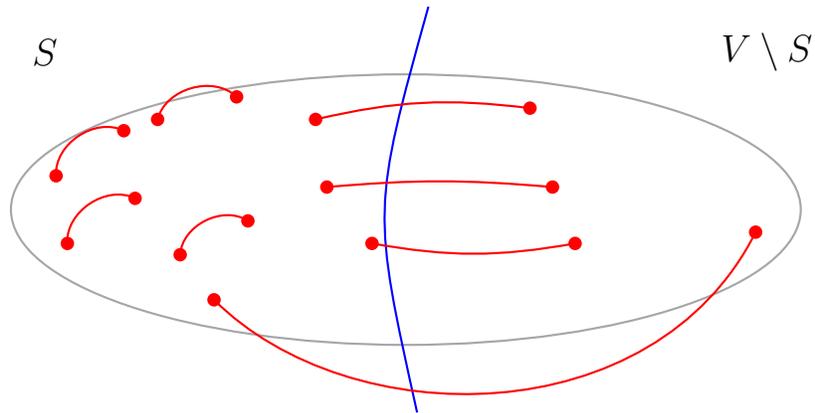Let $T = (N(S^*) \cap N(x)) \cup (N(V \setminus S^*) - x)$.

Note that $|T| \leq \kappa'(G)$, and x and y are in different components of $G - T$, so $\kappa(G) \leq |T|$. We put these inequalities together to obtain $\kappa(G) \leq \kappa'(G)$. $\qquad\square$

4   Edge Cut Size and Node Isolation

Proposition 4.1. *For any $\emptyset \neq S \subsetneq V$,*

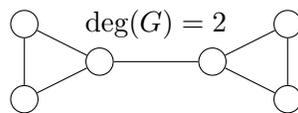$$|\delta(S)| = \sum_{u \in S} deg(u) - 2|E(G[S])|.$$

Proof by picture:

More formally:

Proof. $\sum_{u \in S} \deg(u)$ counts edges in $\delta(S)$ once and edges in $E(G[S])$ twice. $\qquad\square$

If $\kappa'(G) < \deg(G)$ then no smallest edge cut isolates a node. For example, removing the middle edge in this graph does not produce isolated nodes.

In fact, you can use the last proposition to show that if $\delta(S) < \deg(G)$, then $|S| > \deg(G)$.