Lecture 16: Maximum Flow

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In today's lecture, we will start working toward the "max flow - min cut theorem", which is a min-max relation with many applications and corollaries in combinatorics.

1 Flows

Definition 1.1 (Flow). *Let $G = (V, E)$ be a directed multigraph, with source $s \in V$ and target $t \in V$. A flow is an assignment $f_e \in \mathbb{R}_{\geq 0}$ to each $e \in E$ ($f : E \to \mathbb{R}_{\geq 0}$) such that*

$$\sum_{e \in \delta^-(u)} f_e = \sum_{e \in \delta^+(u)} f_e, \qquad \forall\, u \in V,\ u \neq s, t.$$

That is, "mass conservation" holds at every vertex other than the source and sink.

The value of a flow f is denoted by $|f|$ and given by

$$|f| = \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e.$$

1.1 Linear Algebra Formulation of Flows

You can also think of flows through linear algebra (as a vector).

Let $B \in \mathbb{R}^{V \times E}$ be the incidence matrix of G, where

$$B_{we} = \begin{cases} 1, & \text{if } e = (u, v) \text{ and } v = w, \\ -1, & \text{if } e = (u, v) \text{ and } u = w, \\ 0, & \text{otherwise.} \end{cases}$$

For example, consider the following directed graph:

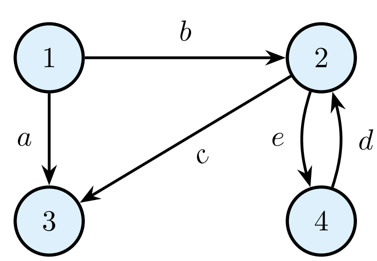


Figure 1: A directed graph with edge set $E = \{a, b, c, d, e\}$.

Its incidence matrix is

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} a & b & c & d & e \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

For $v \in V$, let $\mathbb{1}_v \in \{0,1\}^V$ be its indicator vector. That is,

$$(\mathbb{1}_v)_w = \begin{cases} 1, & \text{if } w = v, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $f \in \mathbb{R}^E_{\geq 0}$ is a flow if and only if

$$Bf = \lambda(\mathbb{1}_t - \mathbb{1}_s) \qquad \text{for some } \lambda \in \mathbb{R}.$$

In this case,

$$|f| = \lambda.$$

This is equivalent to "mass conservation". The vector Bf represents the net inflow at each vertex. In particular,

$$(Bf)_u = 0 \qquad \forall\, u \in V \setminus \{s,t\},$$

and

$$(Bf)_s = -\lambda, \qquad (Bf)_t = \lambda.$$

1.2 Flow Decomposition

Flows can be decomposed as a weighted sum of s-t paths, t-s paths and directed cycles.

For $S \subseteq E$, let $\mathbb{1}_S \in \{0,1\}^E$ be the indicator vector of S, i.e.,

$$(\mathbb{1}_S)_e = \begin{cases} 1, & \text{if } e \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.2. *A vector $f \in \mathbb{R}^E_{\geq 0}$ is an s-t flow if and only if*

$$f = \sum_{i=1}^{k} \lambda_i \mathbb{1}_{S_i},$$

where $S_1, S_2, \ldots, S_k \subseteq E$ is a collection of s-t paths, t-s paths, and directed cycles.

Proof. If S is an s-t path, a t-s path, or a directed cycle, then

$$B\mathbb{1}_S = \lambda(\mathbb{1}_t - \mathbb{1}_s),$$

where

$$\lambda = \begin{cases} 0, & \text{if } S \text{ is a directed cycle,} \\ 1, & \text{if } S \text{ is an } s\text{-}t \text{ path,} \\ -1, & \text{if } S \text{ is a } t\text{-}s \text{ path.} \end{cases}$$

Hence, the weighted sum

$$\sum_{i=1}^{k} \lambda_i \mathbb{1}_{S_i}$$

preserves the identity and, therefore, defines a flow f.

Conversely, if f is a flow, and let

$$E_+(f) = \{e \in E : f_e > 0\}.$$

We can show that $E_+(f)$ contains at least one directed cycle, s-t path, or t-s path, by mass conservation.

For any such $S \subseteq E_+(f)$, let

$$\omega = \min_{e \in S} f_e,$$

and

$$g = f - \omega \cdot \mathbb{1}_S.$$

Then g is again a flow, and moreover

$$E_+(g) \subsetneq E_+(f).$$

So the statement holds inductively. Indeed, since $\omega = \min_{e \in S} f_e$, there exists at least one edge $e_0 \in S$ such that

$$g_{e_0} = f_{e_0} - \omega = 0,$$

so at least one positive edge is removed from the support $E_+(f)$. Therefore the statement follows by induction on $|E_+(f)|$. □

2 Maximum Flows

2.1 Capacitated Directed Multigraphs and Valid Flows

Now, consider $G = (V, E, c)$ be a capacitated directed multigraph, where

$$c : E \to \mathbb{R}_{\geq 0}$$

is a capacity function.

A *valid flow* with source s and sink t is a flow f such that

$$f_e \leq c(e) \qquad \forall e \in E.$$

That is, the capacity constraints are satisfied on every edge.

A *maximum flow* is a valid flow of maximum value.

2.2 Residual Graphs and Augmenting Paths

Question: How do we find a maximum flow?

Idea: Similar to maximum matching, we find augmenting paths.

Let $G = (V, E, c)$ be a capacitated directed multigraph, and let f be a valid flow.

Let $G_f = (V_f, E_f, c_f)$, be the *residual graph* where

- $V_f = V$.
- $E_f = E \cup \overline{E}$, where $\overline{E} = \{\bar{e} = (v, u) \ : \ e = (u, v) \in E\}$.
- The residual capacity function c_f is given by

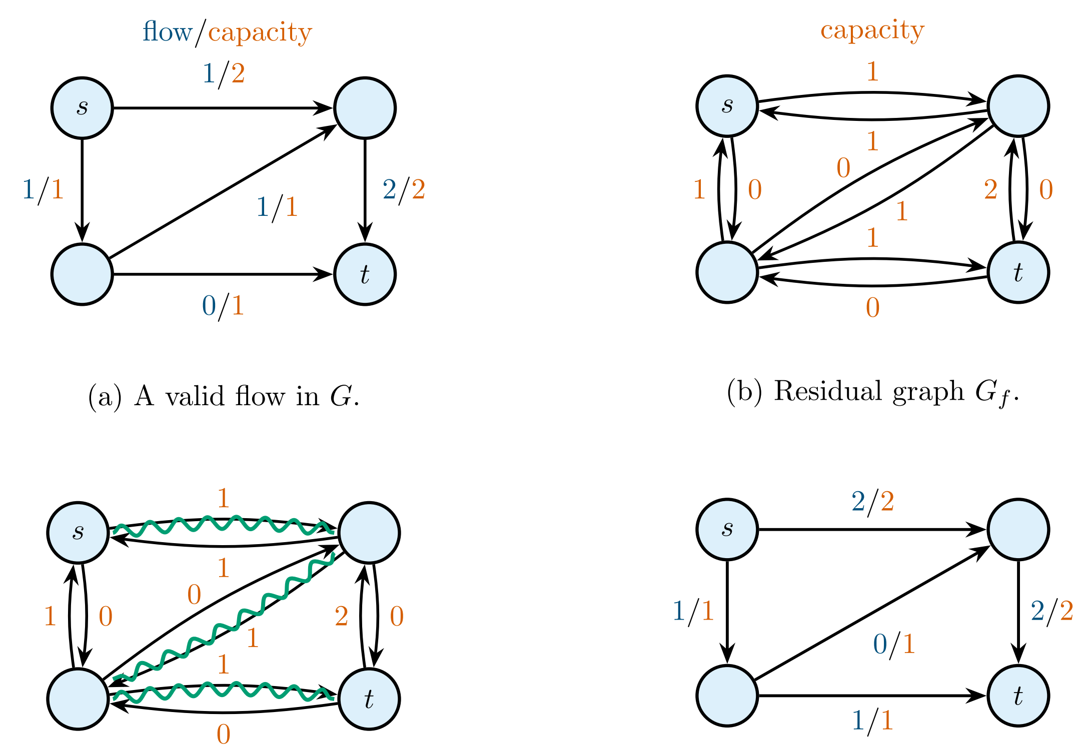
  $$c_f(e) = c(e) - f_e \qquad \text{for } e \in E,$$

  and

  $$c_f(\bar{e}) = f_e \qquad \text{for } \bar{e} \in \overline{E}.$$

An *augmenting path* is a valid s-t path in G_f. Valid means that no arcs with zero capacity are used. The flow value can be increased by augmenting along an s-t path in the residual graph.

Example 2.1.



(a) A valid flow in G.

(b) Residual graph G_f.

(c) An augmenting path in G_f.

(d) Updated flow in G.

Figure 2: Illustration of augmenting a flow: starting from a valid flow in G, constructing the residual graph G_f, finding an augmenting path, and updating the flow accordingly.

Given a valid flow h in G_f, we can define a transformation

$$\pi(h) \in \mathbb{R}^E,$$

where for each edge $e \in E$,

$$(\pi(h))_e = h_e - h_{\bar{e}}.$$

Let $\bar{h} \in \mathbb{R}^E$ be defined by $(\bar{h})_e := h_{\bar{e}},\ e \in E$. Then, $\pi(h) = h - \bar{h}$.

Interpretation:

- Increase flow in e by h_e units.
- Decrease flow in e by $h_{\bar{e}}$ units.

Here h_e denotes the flow sent through the forward residual arc e, and $h_{\bar{e}}$ denotes the flow sent through the backward residual arc $\bar{e}$. Thus, $\pi(h)$ represents the net change of flow on each original edge $e \in E$.

Thus, the different choices of h (flows in the residual graph) and the corresponding $\pi(h)$ encode *all* possible ways of modifying the original flow f, while remaining a valid flow.

Lemma 2.2. *If f is a valid flow in G and h is a valid flow in G_f, then $f + \pi(h)$ is a valid flow in G. Moreover, every valid flow $\tilde{f}$ in G can be written as*

$$\tilde{f} = f + \pi(h)$$

for some valid flow h in G_f.

Proof. Note that $B_{G_f} h = \lambda(\mathbb{1}_t - \mathbb{1}_s)$ for some $\lambda \in \mathbb{R}$.

But also, $B_{G_f} h = B_G h - B_G \overline{h} = B_G(h - \overline{h}) = B_G \pi(h) = \lambda(\mathbb{1}_t - \mathbb{1}_s)$.

Thus, $\pi(h)$ satisfies mass conservation, which implies that $f + \pi(h)$ satisfies mass conservation. In fact, $f + \pi(h)$ is a valid flow in G by our design of c_f.

Conversely, if $\tilde{f}$ is a valid flow in G, let

$$h_e = \max\{\tilde{f}_e - f_e, 0\}, \qquad e \in E,$$

$$h_{\bar{e}} = \max\{f_e - \tilde{f}_e, 0\}, \qquad e \in E.$$

One can verify that h defined in this way is a valid flow in G_f and that $\tilde{f} = f + \pi(h)$. □

Lemma 2.3. *If f is a valid flow in G, then it is a maximum flow if and only if G_f does not contain a valid augmenting path.*

Proof. If f is not a maximum flow, let f^* be a maximum flow. Let h be designed as in previous lemma so that $f^* = f + \pi(h)$, where $|h| > 0$ since f is not maximum. By lemma 1.2, h can be decomposed as s-t paths, t-s paths, and cycles in G_f. At least one part of the decomposition is an s-t path since $|h| > 0$. Thus, G_f contains a valid augmenting path.

Conversely, if P is an augmenting path, let $\omega = \min_{e \in P} c_f(e)$ and $h = \omega \cdot \mathbb{1}_P$, $f + \pi(h)$ is a valid flow with value $|f + \pi(h)| = |f| + \omega > |f|$.

□