Lecture 17: The max flow–min cut theorem.

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1 Review

Last time:

- definition of flows and valid flows;
- an algorithm for finding a maximum flow using augmenting paths in the residual graph G_f.

Recall:

Lemma 1.1. *If f is a valid flow in G, then f is a maximum flow if and only if G_f contains no augmenting s–t path.*

Today we show a min–max relation between s–t flows and edge cuts separating s from t.
We have $G = (V, E, c)$ directed and $s, t \in V$.

Definition 1.2. *An s–t cut is a partition*

$$V = S \sqcup T$$

such that $s \in S$ and $t \in T$. If $e = (u, v)$ is a directed edge, then

$$e \in \delta(S,T) \iff u \in S,\ v \in T.$$

Since $T = V \setminus S$, we also write $\delta(S)$ for $\delta(S,T)$.

Remark 1.3. *Since G is directed, in general $\delta(S,T) \neq \delta(T,S)$.*

Example 1.4. *The two sets $\delta(S,T)$ and $\delta(T,S)$ need not be the same.*

Let

$$c(S,T) = \sum_{e \in \delta(S,T)} c_e,$$

i.e., the total capacity of edges from S to T. Also, for a flow $f \in \mathbb{R}^E_{\geq 0}$, let

$$f(S,T) = \sum_{e \in \delta(S,T)} f_e,$$

i.e., the total flow from S to T.

2 The max flow–min cut theorem

Theorem 2.1 (Max flow–min cut)**.** *For any flow network, the value of any maximum flow is equal to the capacity of any minimum cut.*

Proof. Let (S, T) be any s–t cut. Let $1_T \in \mathbb{R}^V$ be such that

$$(1_T)_u = \begin{cases} 1, & u \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Recall the incidence matrix $B \in \mathbb{R}^{V \times E}$. Put

$$x = (1_T)^T B.$$

Then

$$x_e = \begin{cases} 1, & e \in \delta(S, T), \\ -1, & e \in \delta(T, S), \\ 0, & \text{otherwise.} \end{cases}$$

We first show that the value of any s–t flow is upper bounded by the value of any s–t cut. In particular,

$$\text{max flow value} \leq \text{min cut capacity}.$$

Let $f \in \mathbb{R}^E_{\geq 0}$ be a valid s–t flow. Then

$$\begin{aligned} (1_T)^T B f &= x^T f \\ &= f(S, T) - f(T, S) \\ &\leq f(S, T) \qquad \text{since } f \in \mathbb{R}^E_{\geq 0}, \\ &\leq c(S, T) \qquad \text{since } f \text{ is a valid flow.} \end{aligned}$$

Also,

$$(1_T)^T B f = (1_T)^T (1_t - 1_s) \, |f| = |f|.$$

Therefore,

$$|f| \leq c(S, T).$$

Now note where the two inequalities are tight:

- $f(S, T) - f(T, S) \leq f(S, T)$ is tight if and only if $f(T, S) = 0$, i.e. $f_e = 0$ for every $e \in \delta(T, S)$;
- $f(S, T) \leq c(S, T)$ is tight if and only if $f(S, T) = c(S, T)$, i.e. $f_e = c_e$ for every $e \in \delta(S, T)$.

We now show the converse. Suppose f is a maximum flow. Then there is no s–t path with positive residual capacity in G_f. Partition

$$V = S \sqcup T$$

where S is the set of vertices reachable from s in G_f and $T = V \setminus S$. Then $s \in S$ and $t \in T$, so (S, T) is an s–t cut. Also, G_f contains no edge from S to T with positive residual capacity.

Hence, for every $e = (u, v) \in \delta(S, T)$,

$$c_f(e) = c_e - f_e = 0,$$

so $f_e = c_e$. This makes the second inequality tight.

Now take $e = (u, v) \in \delta(T, S)$. Then $u \in T$ and $v \in S$, so the reverse residual edge (v, u) goes from S to T. Since no edge from S to T has positive residual capacity in G_f, the residual capacity of (v, u) is 0. But that residual capacity is exactly f_e. Hence $f_e = 0$, and the first inequality is tight as well.

Therefore, for this choice of maximum flow f and cut (S, T),

$$|f| = c(S, T).$$

Since every flow value is at most every cut capacity, this cut is a minimum cut. □

3 Applications with ∞-capacity edges

We now cover applications of the theorem. For this it is useful to consider flow networks where

$$c : E \to \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Fact 3.1.

- *If G contains an s–t path made entirely of ∞-capacity edges, then the maximum flow is unbounded.*
- *Otherwise, the maximum flow value is finite, and the max flow–min cut theorem still applies.*

So in these applications we may design gadgets with ∞-capacity edges as needed, as long as we do not introduce an ∞-capacity s–t path.

4 König–Egerváry

Let us revisit something old.

Theorem 4.1 (König–Egerváry)**.** *If G is a bipartite undirected graph, then the cardinality of a maximum matching equals the cardinality of a minimum vertex cover.*

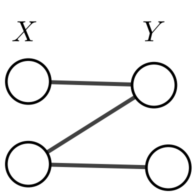
Proof. Our gadget is a flow network G' with a super-source s and a super-target t.

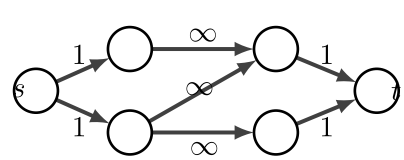
Given $G = (X \sqcup Y, E)$, we let

- s be connected to all of X;
- all edges in E be oriented from X to Y;
- all of Y be connected to t.

Set the capacities by

$$c(s, x) = 1 \quad (x \in X), \qquad c(y, t) = 1 \quad (y \in Y), \qquad c(x, y) = \infty \quad ((x, y) \in E).$$





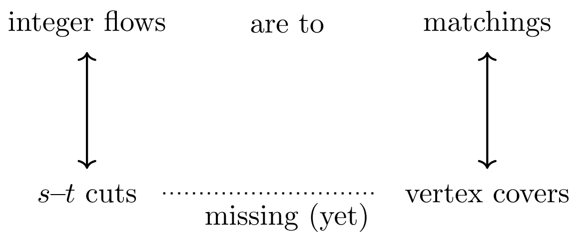
If we want only finite capacities, we may replace each ∞ by any integer $M > |X| + |Y|$; the same cuts remain optimal. In that version all capacities are integral. However, having ∞ capacities will make proofs simpler.

Any integer flow f in G' gives a matching M_f in G by

$$\{x, y\} \in M_f \quad \iff \quad f(x, y) = 1.$$

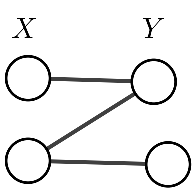
Similarly, any matching in G gives an integer flow in G' of the same value. So maximum matching size is the same as maximum integer flow value.

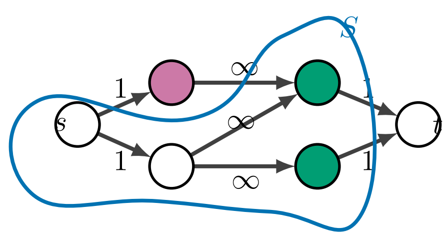
At this point we have



Now consider any s–t cut $(S, T) = (S, V \setminus S)$ of finite capacity, and let

$$A = (X \cap T) \cup (Y \cap S) = (X \cap (V \setminus S)) \cup (Y \cap S).$$





We claim that A is a vertex cover. Indeed, if $(x, y) \in E$ has no endpoint in A, then

$$x \notin X \cap T \quad \text{and} \quad y \notin Y \cap S.$$

Since $x \in X$ and $y \in Y$, this means

$$x \in X \cap S, \qquad y \in Y \cap T.$$

So the edge (x, y) crosses from S to T, and in our gadget it has capacity ∞. This contradicts the assumption that (S, T) has finite capacity.

Now the cut capacity $c(S, T)$ is exactly the number of edges

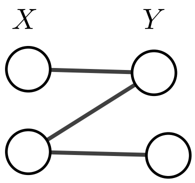
- from s to $X \cap (V \setminus S)$,
- from $Y \cap S$ to t.

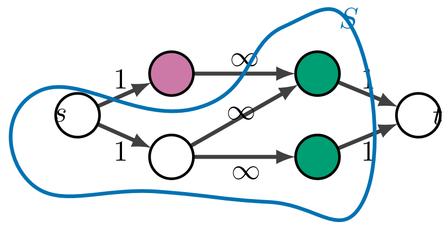
Hence

$$c(S, T) = |A|.$$

Conversely, any vertex cover A gives an s–t cut of capacity $|A|$ by the reverse construction

$$S = \{s\} \cup (X \setminus A) \cup (Y \cap A).$$





So

$$\text{max flow value} = \text{max matching size}$$
$$\Updownarrow \text{ max flow–min cut}$$
$$\text{min cut value} = \text{min vertex cover size}$$

Therefore,

$$\text{maximum matching size} = \text{minimum vertex cover size}.$$

□

Remark 4.2. *Idea: min–max relation, duality, and the right gadget.*