Lecture 2: More on Graphs

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1 Degree

1.1 Undirected Graphs

For an undirected graph $G = (V, E)$, for each $u \in V$, $\deg(u)$ is the number of edges adjacent to it. Formally,

$$\deg(u) := |\{e = \{v, w\} \in E : u \in e\}|.$$

Note the following terminology:

- If $\deg(u) = 1$, u is a leaf.
- If $\deg(u) = 0$, u is isolated.

1.2 Directed Graphs

For a directed graph, there are two different notions of degree. The in-degree (out-degree) of $u \in V$ is the number of edges going into (out of) u. Formally,

$$\deg^-(u) := |\{e = (v, w) \in E : w = u\}|$$

and

$$\deg^+(u) := |\{e = (v, w) \in E : v = u\}|.$$

1.3 Examples

Consider the undirected graph in Figure 1. Then, we have

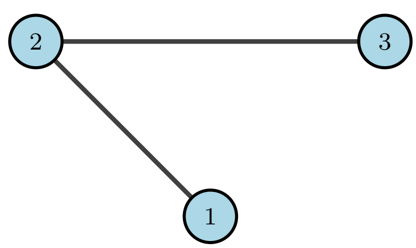

Figure 1: An undirected graph.

- $\deg(1) = 1$,
- $\deg(2) = 2$, and
- $\deg(3) = 1$.

Similarly, consider the directed graph in Figure 2. Then, we have

- $\deg^-(1) = 0$ and $\deg^+(1) = 1$,
- $\deg^-(2) = 1$ and $\deg^+(2) = 1$, and
- $\deg^-(3) = 1$ and $\deg^+(3) = 0$.

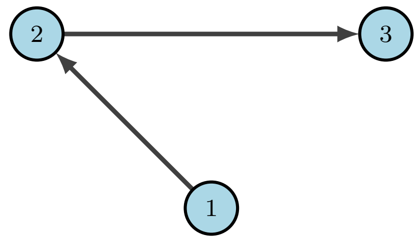

Figure 2: A directed graph.

2 Subgraphs

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$. H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

3 Union of Graphs

The union of $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is a graph $W = (V(W), E(W))$ such that

- $V(W) = V(G) \cup V(H)$ and
- $E(W) = E(G) \cup E(H)$.

4 Paths

4.1 Undirected Graphs

A path is a graph $P = (V(P), E(P))$ where $V(P)$ can be totally ordered such that two nodes are adjacent in P only if they are adjacent in the order.

We can visualize this in Figure 3 and Figure 4.

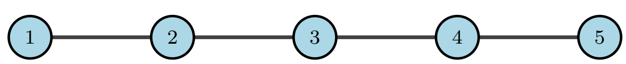

Figure 3: A path.

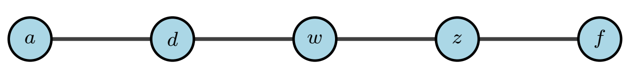

Figure 4: Another path.

In the example, both graphs are paths. The graph in Figure 4 is a a path: to see this algebraically, we can map each node to $\mathbb{N}$ where a maps to 1, d maps to 2, w maps to 3, and so on. A path can be expressed as:

- A sequence of nodes, such as $v_1, v_2, \ldots, v_k$.
- A sequence of edges, such as $e_1, e_2, \ldots, e_{k-1}$.
- A sequence of nodes and edges, such as $v_1, e_1, v_2, e_2, \ldots, v_k, e_{k-1}$.

Note that $|E(P)| = |V(P)| - 1$. If $v_1 = s$ and $v_k = t$, we say P is an $\{s,t\}-$path. If P is an $\{s,t\}-$path P, then

- $\deg(s) = 1$,
- $\deg(t) = 1$, and
- $\deg(u) = 2$ for all $u \in V(P)$ with $u \neq s, t$.

4.2 Directed Graphs

Directed paths have the same formal definition with the additional requirement that

$$e_i = (v_i, v_{i+1})$$

for all $i \in [k-1]$. In other words, the head of an edge is the same as the tail of the subsequent edge. Figure 5 shows a directed path.

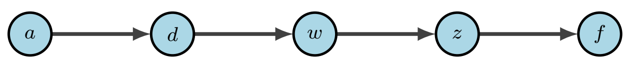

Figure 5: A directed path.

Note that again $|E(P)| = |V(P)| - 1$. If P is an (s,t)-path, then

- $\deg^-(s) = 0$ and $\deg^+(s) = 1$,
- $\deg^-(t) = 1$ and $\deg^+(t) = 0$, and
- $\deg^-(u) = \deg^+(u) = 1$ for all $u \in V(P)$ with $u \neq s, t$.

5 Cycles

5.1 Undirected Graphs

A cycle is a graph $C = (V(C), E(C))$ such that its nodes can be placed around a circle on the plane with two nodes are adjacent on the circle if and only if they are adjacent in C. Figure 6 shows a cycle.

Alternatively, a cycle C is a path for which we connect its endpoints with an edge. Note that $|E(C)| = |V(C)|$ and $\deg(u) = 2$ for all $u \in V(C)$.

5.2 Directed Graphs

Directed cycles have the same formal definition with the additional requirement of respecting directionality.

6 Connectivity

6.1 Undirected Graphs

Let $G = (V, E)$ be an undirected graph. Two (unordered) nodes $u, v \in V$ are connected if G contains a $\{u,v\}$-path. The graph G is connected if all (unordered) pairs $u, v \in V$ are connected. Figure 7 shows a connected graph with multiple $\{u,v\}$-paths.

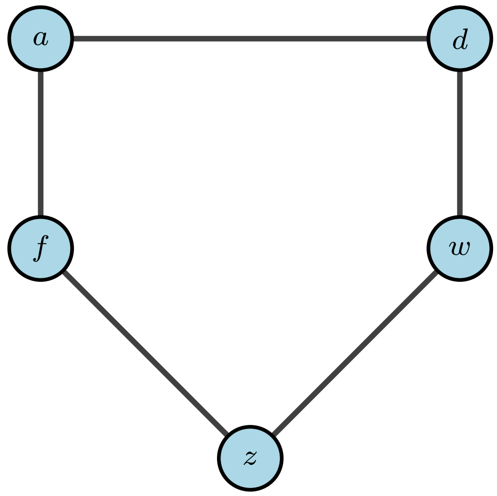

Figure 6: A cycle.

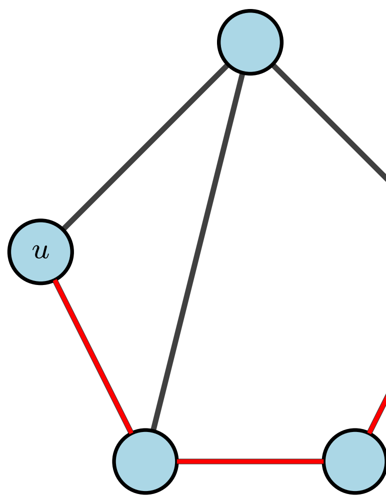

Figure 7: A connected graph.

6.2 Directed Graphs

For a directed graph $G = (V, E)$, there are two notions of connectivity:

- G is weakly connected if its corresponding undirected graph is connected. For example, the graph in Figure 5 is weakly connected.
- G is strongly connected if it contains a (u, v)-path for all ordered pairs $u, v \in V$.

7 Connected Components

7.1 Undirected Graphs

A connected component of a graph $G = (V, E)$ is an inclusion-wise maximal connected subgraph of G. Here, inclusion-wise maximal means there does not exist another other subgraph with the same property (i.e., connectivity) that strictly contains it. For example, Figure 8 shows a graph with three connected components, of sizes 4, 3, and 1 from left to right.

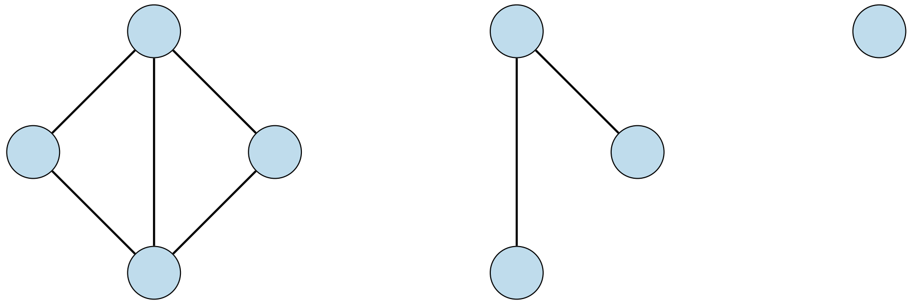

Figure 8: A graph with three connected components.

7.2 Directed Graphs

A weakly connected component of a directed graph $G = (V, E)$ is an inclusion-wise maximal weakly connected subgraph of G. Similarly, a strongly connected component of G is an inclusion-wise maximal strongly connected subgraph of G.