

**Lecture 20:** More on Planer Graphs.

**Date:** April 8, 2026

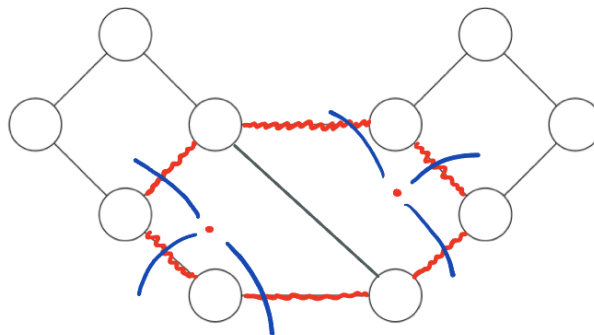
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We will start with a duality theorem w.r.t cycles in a plane graph and bonds (i.e., minimal edge cuts) in its dual.

**Theorem 1.** *Edges in a plane graph  $G$  form a cycle if and only if the corresponding edges in  $G^*$  form a bond.*

*Proof.* Let  $D \subseteq E(G)$  and consider its corresponding set  $D^* \subseteq E(G^*)$ .

$\Rightarrow$  Suppose  $D$  forms a cycle in  $G$ . Let  $S$  be the faces of  $G$  inside of  $D$  ( $S \neq \phi$ , because of the Jordan Curve Theorem). Then, the edge cut  $\delta(S)$  in  $G^*$  consists of all the edges of  $G^*$  that cross edges of  $D$ , i.e.,  $D^*$ .



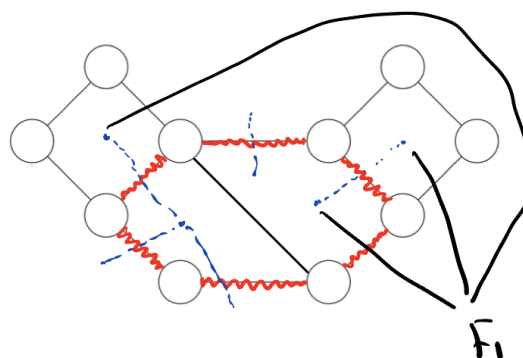
$D$  forms a cycle.

$D^*$  is an edge cut in the dual, i.e.,  $D^* = \delta(S)$ .

Conversely, suppose  $D$  contains no cycle in  $G$ .

$\Rightarrow D$  encloses no region.

The unbounded face of  $G$  is reachable from any where without crossing  $D$ .



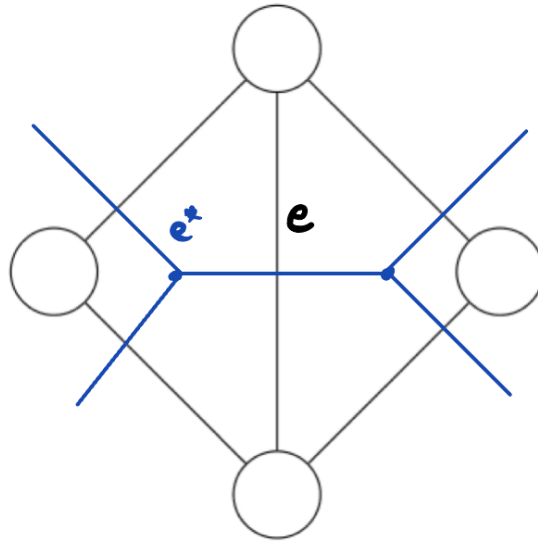
$D$  does not contain a cycle (imagine the diagonal edge on the bottom right is not there).

$\Rightarrow G^* - D^*$  is connected.

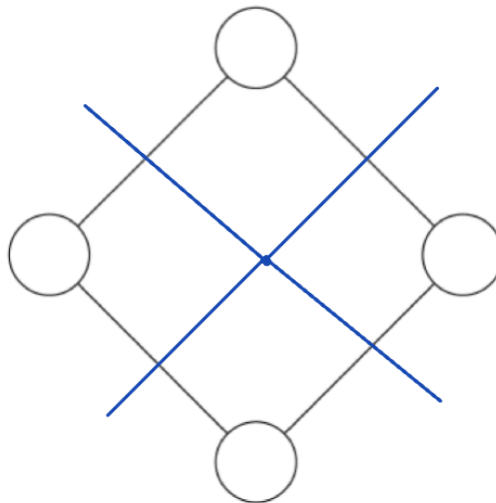
$\Rightarrow D^*$  is not an edge cut.

Hence, when  $D$  is a cycle,  $D^*$  is a minimal edge cut, i.e.,  $D^*$  is a bond. □

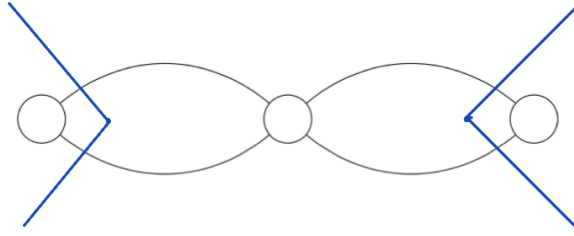
There is also duality w.r.t deletion and contraction.



The original graph with primal edge  $e$  and dual edge  $e^*$



We deleted  $e$  and contracted  $e^*$ .



We contracted  $e$  and deleted  $e^*$

Euler's Formula the # of nodes, edges, and faces of a plane graph.

**Theorem 2** (Euler's Formula). *Let  $G = (V, E)$  be a connected plane graph with  $|V| = n$ ,  $|E| = m$ , and  $r$  faces. Then  $n - m + r = 2$ .*

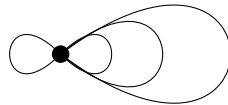
*Proof.* By induction on  $n$ .

Base Case ( $n = 1$ ):

$\hookrightarrow$  If  $m = 0 \Rightarrow r = 1$

$$\Rightarrow n - m + r = 1 - 0 + 1 = 2.$$

$\hookrightarrow$  Otherwise, since  $n = 1, m > 0$ ,  $G$  looks like a bouquet of loops.



Each new loop takes a face and splits it into two faces.  $\Rightarrow m$  loops,  $r = m + 1$

$$\Rightarrow n - m + r = n - m + m + 1 = 1 + 1 = 2.$$

Induction step ( $n > 1$ ): since  $G$  is connected and  $n > 1$ , there exists some edge  $e$  that is not a loop contract  $e$ , to obtain a new plane graph  $G'$  with  $n' = n - 1$ ,  $m' = m - 1$ , and  $r' = r$  since contracting an edge shortens the boundary of a face but does not remove it. By the inductive hypothesis:

$$\begin{aligned} 2 &= n' - m' + r' = n' + 1 - (m' + 1) + r' \\ &= n - m + r \end{aligned}$$

□

**Corollary 3.** *All plane embeddings of planar graph have the same number of faces.*

*Proof.*  $n, m$  and 2 are fixed  $\Rightarrow r$  is fixed.

□

We can use Euler's Formula as a sufficient condition for non planarity.

**Theorem 4.** *Let  $G$  be a simple connected planar graph, with  $n \geq 3$  nodes, and  $m$  edges. Then:*

1.  $m \leq 3n - 6$ .
2. If  $G$  is triangle-free (i.e., no 3-cycles),  $m \leq 2n - 4$

*Proof.* 1. If  $n \geq 3, \ell(F_i) \geq 3$  for all faces of its plane embedding. By the dual degree sum formula,  $2m = \sum_{i=1} \ell(F_i) \geq 3r$ .

$$\Rightarrow r \leq \frac{2m}{3} \quad \Rightarrow m \leq 3n - 6$$

$$n - m + r$$

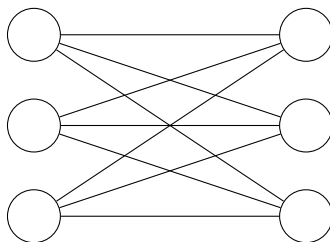
2. If  $G$  is triangle-free,  $\ell(F_i) \geq 4, \forall F_i$

$$2m \geq 4r$$

$$n - m + r = 2 \quad \Rightarrow m \leq 2n - 4$$

□

As an application, we can show that  $K_5$  is not planar. As another application, we can show that the complete bipartite graph  $K_{3,3}$  is not planar.



$K_{3,3}$

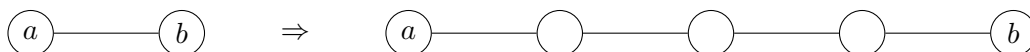
1.  $n = 6$ ,
2.  $m = 3 \cdot 3 = 9$ ,
3.  $K_{3,3}$  is triangle free, because it is bipartite (all cycles have even length).

$$m = 9 \not\leq 2n - 4 = 2 \cdot 6 - 4 = 8$$

So far, we know that  $K_5$  and  $K_{3,3}$  are not planar. In fact, these are the main ingredients that lead to non planarity.

A subdivision of a graph  $G$  is a graph obtained from  $G$  by subdividing its edges with new nodes from paths.

**Example**



**Proposition 5.** *If a graph  $G$  Contains a subdivision of  $K_5$  or  $K_{3,3}$ , then  $G$  is not planar.*

*Proof.* A subgraph of a planar graph is planar. Therefore, it suffices to show that a subdivision of  $K_5$  or  $K_{3,3}$  is not planar. But  $K_5$  and  $K_{3,3}$  are not planar, and subdividing their edges has no effect on their planarity / non planarity.  $\square$

In fact, this is also sufficient.

**Theorem 6** (Kuratowski). *A graph is planar if and only if it does not contain a subdivision of  $K_5$  or  $K_{3,3}$*

**Note:** This theorem does not directly yield an efficient planarity testing algorithm: there are exponentially - many subgraph to check. However, there exists polynomial-time algorithms to test planarity.