Lecture 21**:** Coloring Planar Graphs

Date: April 13, 2026 **Scribe:** Mark Johnson

1 Greedy Coloring

For a graph G_1, the **greedy coloring** w.r.t. an ordering $v_1, v_2, ..., v_n$ of its nodes, colors each node in that order and gives v_i the least index color not used by its earlier-colored neighbors.

- This need not produce an optimal coloring, but whatever coloring it produces gives an upper-bound to the $\chi(G)$.

Proposition: $\chi(G) \leq \triangle(G) + 1$, where $\triangle(G)$ is the maximum degree in G.

Proof

In any ordering of $V(G)$, each node has at most $\triangle(G)$ earlier-colored neighbors. This implies that the Greedy Algorithm will use no more than $\triangle(G) + 1$ colors.

□

A graph is **k-degenerate** if every subgraph has a node of degree at most k.

A **smallest-last ordering** of V is constructed iteratively by letting v_i be a node of minimum degree in $G - \{v_1, v_2, ..., v_{i-1}\}$

Proposition (Szkeras-Wilf): If G is k-degenerate, then G is $(k+1)$-colorable.

Proof

If G is k-degenerate, then the smallest-last ordering gives each node at most k neighbors among the earlier-colored nodes. This implies that the greedy coloring w.r.t. this ordering used at most $k + 1$ colors.

□

2 How Does This Relate to Planar Graphs?

From lecture 20, we know that a planar graph with n nodes and m edges satisfies $m \leq 3n - 6$ (this followed from Euler's formula). Therefore, planar graphs are 5-degenerate.

$$\sum_{u \in V} \deg(u) = 2m \leq 6n - 12$$

$$\frac{1}{n} \sum_{u \in V} \deg(u) \leq \frac{6n - 12}{n} = 6 - \frac{12}{n} < 6$$

$\implies$ the average node has degree < 6, but degree is an integer quantity.
$\implies$ for every planar graph, there must exist a node of degree ≤ 5.

Together with our proposition (Szkeras-Wilf), we find that planar graphs are 6-colorable.

3 A Stronger Result

3.1 Setup

A graph G is **color-critical** if $X(H) < \chi(G)$ for all subgraphs $H \subseteq G$. If it also holds that $\chi(G) = K$, we say that G is K-critical.

Proposition If G is K-critical, then $\delta(G) \geq K - 1$, where $\delta(G)$ is the minimum degree of G.

Proof

If G is K-critical, then $\chi(G - u) = K - 1$ for all $u \in V$.

Now consider any $u \in V$. Since $\chi(G - u) = K - 1$, there exists a proper coloring of $G - u$ with exactly $K - 1$ colors (by definition of χ).

If $\deg(u) < K - 1$ in G, then $N(u)$ uses $\leq K - 2$ colors.

Therefore, at least one of the $K - 1$ colors is not used to color $N(u)$.

This implies this color can be used to color u to produce a proper coloring of G with $K - 1$ colors. This is a contradiction to the assumption that $\chi(G) = K$.

$\square$

3.2 Five Color Theorem

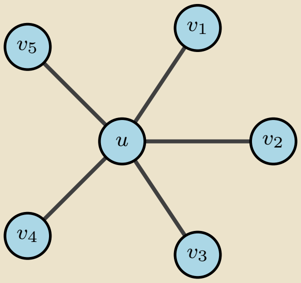
Theorem (Five Color Theorem): Every planar graph is five-colorable.

Proof

Let G be a planar graph. Subgraphs of planar graphs are planar. Therefore it's sufficient to show there are no 6-critical planar graphs.

BWOC, suppose G is 6-critical. By the above proposition, $\delta(G) \geq 6 - 1 = 5$.

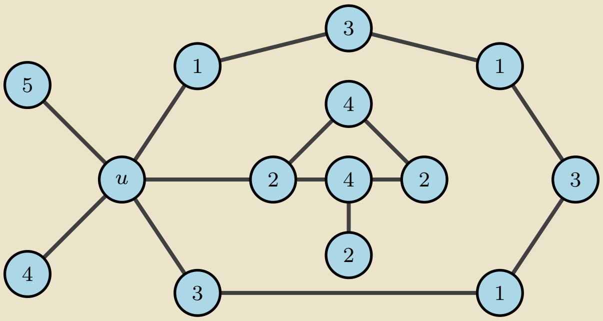
Let $u \in V$ s.t. $\deg(u) = 5$.
Let $c : V - u \to [5]$ be a proper coloring of $G - u$. Then all colors in $[5]$ are used to color $N(u)$.
Let $v_1, v_2, ..., v_5$ be the neighbors of u in clockwise order and wlog set $c(v_i) = i$



(Note that there could be other objects in the graph)

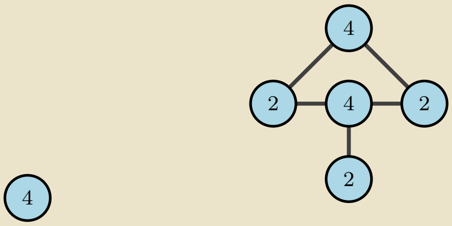
Let G_{ij} be the subgraph of $G - u$ induced by the nodes with colors i and j.

Now consider any component of G_{ij}. If we switch its colors (i.e. nodes with color i are given color j and vice-versa, this produces another 5-coloring of $G - u$.

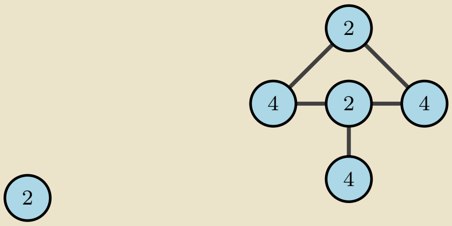


Continued on next page...

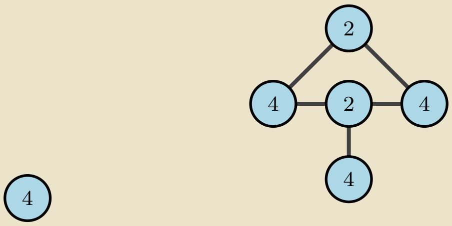
For example, below is G_{24}



If we swap 2 and 4 in G_{24}, then that produces another 5-coloring of the graph.

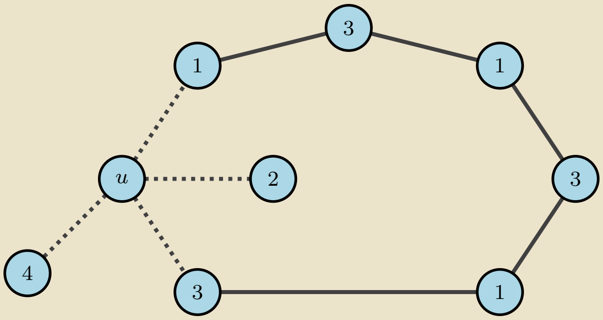


Now consider the connected component of G_{ij} containing node v_i. If this component does not contain v_j, then switching its colors removes color i from $N(u)$.



But then we can color node u with color i to produce a proper 5-coloring of G. Therefore G is 5-colorable (a contradiction) unless the component of G_{ij} containing v_i also contains v_j for each i, j.

Let P_{ij} be a path from v_i to v_j in G_{ij} this path can be completed into a cycle using node u.



This cycle C separates two other nodes. In the picture, C separates v_2 from v_4. By Jordan Curve Theorem, P_{24} must cross C. Since G is planar, this crossing must take place at shared nodes, but the nodes of P_{13} have colors 1 or 3 and the nodes of P_{24} have colors 2 or 4. Therefore, P_{24} and P_{13} have no shared node, a contradiction, so G cannot be 6 critical. Therefore every planar graph is 5-colorable.

□

This result was strengthened even further in 1976:

Theorem (Appel, Haken, Koch): Every planar graph is 4-colorable.

This was the first major example of a computer-aided proof for case analysis.