Lecture 22: The Graph Laplacian

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1 Spanning Trees

Definition 1.1. *A subgraph $H \subseteq G$ is* spanning *if $V(H) = V(G)$.*

Definition 1.2. *A* spanning tree *of G is a subgraph T that is both spanning and a tree. Let*

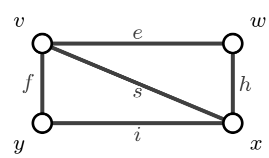
$$\mathcal{ST}(G)$$

be the set of spanning trees of G.

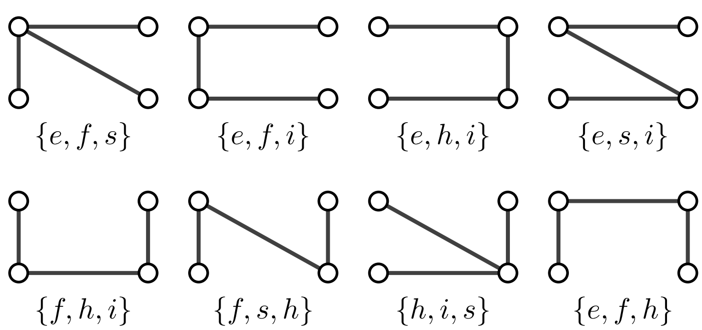
Example 1.3. *Consider the graph G shown below, with vertex set $\{v, w, x, y\}$ and edge set*

$$E(G) = \{e, f, s, h, i\},$$

where $e = \{v, w\}$, $f = \{v, y\}$, $s = \{v, x\}$, $h = \{w, x\}$, and $i = \{y, x\}$.



Then $\mathcal{ST}(G)$ consists of the following eight spanning trees:



Thus

$$|\mathcal{ST}(G)| = 8.$$

The general goal for the next few lectures is to derive a formula for $|\mathcal{ST}(G)|$. If $G = K_n$, then we know from quite a few lectures ago that

$$|\mathcal{ST}(K_n)| = n^{n-2}.$$

But we want a more general formula.

2 Matrices Associated to an Undirected Graph

Definition 2.1. *Let $G = (V, E)$ be an undirected graph. Its* adjacency matrix *is*

$$A(G) = A \in \mathbb{R}^{V \times V},$$

where

$$a_{uv} = \begin{cases} 1, & \text{if } \{u, v\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

We must keep a consistent labeling of the rows and columns; that is, we fix an ordering of V and stick to it. The matrix A is symmetric.

For the graph G above, using the vertex order (v, w, x, y), the adjacency matrix is

$$A(G) = \begin{array}{c|cccc} & v & w & x & y \\ \hline v & 0 & 1 & 1 & 1 \\ w & 1 & 0 & 1 & 0 \\ x & 1 & 1 & 0 & 1 \\ y & 1 & 0 & 1 & 0 \end{array}.$$

Definition 2.2. *The* undirected incidence matrix *of G is*

$$B(G) = B \in \mathbb{R}^{V \times E},$$

where

$$b_{v,e} = \begin{cases} 1, & \text{if } v \text{ is an endpoint of } e, \\ 0, & \text{otherwise.} \end{cases}$$

This differs slightly from the incidence matrix for directed graphs.

Using the vertex order (v, w, x, y) and the edge order (e, f, s, h, i), the undirected incidence matrix for the example is

$$B(G) = \begin{array}{c|ccccc} & e & f & s & h & i \\ \hline v & 1 & 1 & 1 & 0 & 0 \\ w & 1 & 0 & 0 & 1 & 0 \\ x & 0 & 0 & 1 & 1 & 1 \\ y & 0 & 1 & 0 & 0 & 1 \end{array}.$$

Definition 2.3. *The* degree matrix *of G is the diagonal matrix*

$$C(G) = C \in \mathbb{R}^{V \times V}$$

whose diagonal entries are

$$c_{uu} = \deg(u).$$

For the example,

$$C(G) = \begin{array}{c|cccc} & v & w & x & y \\ \hline v & 3 & 0 & 0 & 0 \\ w & 0 & 2 & 0 & 0 \\ x & 0 & 0 & 3 & 0 \\ y & 0 & 0 & 0 & 2 \end{array}.$$

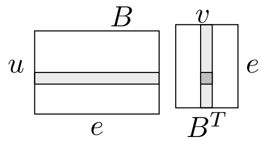
The matrices A, B, and C are nicely related.

Proposition 2.4. *For an undirected graph G,*

$$B(G)B(G)^T = A(G) + C(G).$$

Proof. Recall that $B \in \mathbb{R}^{V \times E}$, so $BB^T \in \mathbb{R}^{V \times V}$. Thus BB^T is indexed by pairs of vertices.

For $u, v \in V$, the entry $(BB^T)_{uv}$ is obtained by taking the dot product of the u-row of B with the v-row of B:



$$(BB^T)_{uv} = \sum_{e \in E} b_{u,e} b_{v,e}.$$

If $u = v$, then

$$(BB^T)_{uu} = \sum_{e \in E} b_{u,e}^2 = \sum_{e \in E} \mathbf{1}\{u \text{ is an endpoint of } e\} = \deg(u) = C_{uu}.$$

If $u \neq v$, then

$$\begin{aligned}(BB^T)_{uv} &= \sum_{e \in E} b_{u,e} b_{v,e} \\ &= \sum_{e \in E} \mathbf{1}\{u \text{ is an endpoint of } e\}\mathbf{1}\{v \text{ is an endpoint of } e\} \\ &= \mathbf{1}\{\{u, v\} \in E\} = A_{uv}.\end{aligned}$$

Therefore the diagonal entries of BB^T agree with C, and the off-diagonal entries agree with A. Hence $BB^T = A + C$. $\square$

3 Orientations and Directed Incidence Matrices

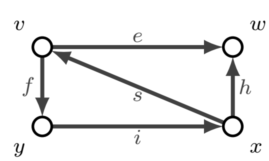
We will need a directed version of G going forward.

Definition 3.1. *An* orientation *of an undirected graph G is a directed graph D such that*

$$V(D) = V(G),$$

and for each edge $\{u, v\} \in E(G)$, exactly one of (u, v) or (v, u) is in $E(D)$. We say that G is the underlying undirected graph of D.

For the example, one orientation D is



so the directed edges are $e = (v, w)$, $f = (v, y)$, $s = (x, v)$, $h = (x, w)$, and $i = (y, x)$.

Definition 3.2. *Recall the incidence matrix for a directed graph D:*

$$B(D) = B \in \mathbb{R}^{V \times E},$$

where

$$b_{v,e} = \begin{cases} 1, & \text{if } e = (u,v) \text{ for some } u, \\ -1, & \text{if } e = (v,u) \text{ for some } u, \\ 0, & \text{otherwise.} \end{cases}$$

For the oriented graph D above,

$$B(D) = \begin{array}{c|ccccc} & e & f & s & h & i \\ \hline v & -1 & -1 & 1 & 0 & 0 \\ w & 1 & 0 & 0 & 1 & 0 \\ x & 0 & 0 & -1 & -1 & 1 \\ y & 0 & 1 & 0 & 0 & -1 \end{array}.$$

Here are some properties of D.

Proposition 3.3. *Let G be an undirected graph and let D be an orientation of G. Then:*

(a) If the rows of $B(D)$ are $b_1, b_2, \ldots, b_n$, then

$$b_1 + b_2 + \cdots + b_n = \mathbf{0}.$$

(b) We have

$$B(D)B(D)^T = C(G) - A(G).$$

Proof. For part (a), each column of $B(D)$ has exactly one 1, exactly one -1, and zeros everywhere else. Hence the sum of the rows is the zero vector.

For part (b), use the same case analysis as in the previous proof. If $u = v$, then

$$(B(D)B(D)^T)_{uu} = \sum_{e \in E} b_{u,e}^2 = \deg(u) = C_{uu}.$$

If $u \neq v$, then a nonzero contribution can occur only when u and v are endpoints of the same edge. In that case one endpoint contributes 1 and the other contributes -1, so

$$(B(D)B(D)^T)_{uv} = -\mathbf{1}\{\{u,v\} \in E(G)\} = -A_{uv}.$$

Therefore $B(D)B(D)^T = C(G) - A(G)$. □

In part (b), the left-hand side is computed from an arbitrary orientation D, but the right-hand side only depends on the underlying undirected graph. Therefore $B(D)B(D)^T$ is independent of the choice of orientation. This matrix is extremely useful.

4 The Graph Laplacian

Definition 4.1. *The* Laplacian matrix *of an undirected graph G is*

$$L(G) = C(G) - A(G).$$

For the example,

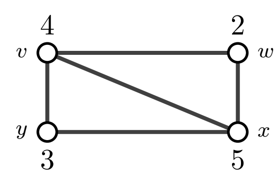
$$L(G) = \begin{array}{c|cccc} & v & w & x & y \\ \hline v & 3 & 0 & 0 & 0 \\ w & 0 & 2 & 0 & 0 \\ x & 0 & 0 & 3 & 0 \\ y & 0 & 0 & 0 & 2 \end{array} - \begin{array}{c|cccc} & v & w & x & y \\ \hline v & 0 & 1 & 1 & 1 \\ w & 1 & 0 & 1 & 0 \\ x & 1 & 1 & 0 & 1 \\ y & 1 & 0 & 1 & 0 \end{array} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

The sum of any row of $L(G)$ is zero: there is one $\deg(u)$ entry, $\deg(u)$ many -1's, and zeros everywhere else.

For example,

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \\ -3 \end{bmatrix}.$$

The corresponding labeled graph is



In particular, since every row sum of $L(G)$ is zero,

$$L(G)\mathbf{1} = \mathbf{0}.$$

Thus the matrix has linearly dependent columns, so

$$\det(L(G)) = 0.$$

Now let $L_{i,j}$ be the submatrix of L obtained by removing the i-th row and the j-th column. For the example above,

$$L_{1,1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad L_{1,3} = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix}.$$

Their determinants are

$$\det(L_{1,1}) = 2\big(3 \cdot 2 - (-1)(-1)\big) - (-1)\big((-1)(2) - 0\big) = 8,$$

and

$$\det(L_{1,3}) = -1\big((-1)(2)\big) - 2\big((-1)(2) - (-1)(-1)\big) = 8.$$

Therefore

$$\det(L_{1,1}) = \det(L_{1,3}) = 8 = |\mathcal{ST}(G)|.$$

Next time, we will show that this is not a coincidence.