Lecture 23: The Matrix–Tree Theorem

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In this lecture, we study the Matrix–Tree Theorem. To prepare for its proof, we first introduce notation for submatrices and cofactors, and then recall the Cauchy–Binet formula.

1 Submatrix and Cofactor

Let M be a matrix indexed by $S \times T$. If $I \subseteq S$ and $J \subseteq T$, let

$$M_{I,J}$$

be the submatrix of M whose rows are indexed by I and whose columns are indexed by J.

If

$$I = S - \{s\} \qquad \text{and} \qquad J = T - \{t\},$$

then

$$M_{\hat{s},\hat{t}} = M_{I,J}.$$

If M is a square matrix, meaning $S = T = [n]$, recall that the (i,j)-cofactor of M is

$$m_{\hat{i}\hat{j}} = (-1)^{i+j} \det M_{\hat{i},\hat{j}}.$$

This notation will be useful later.

2 The Cauchy–Binet Formula

Theorem 2.1 (Cauchy–Binet)**.** *Let $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times m}$. Then*

$$\det(QR) = \sum_{K \in \binom{[n]}{m}} \det Q_{[m],K} \cdot \det R_{K,[m]}.$$

As noted in class, if $m = n$, this recovers the identity

$$\det(QR) = \det Q \cdot \det R.$$

Before proving Cauchy–Binet, we recall that for a square matrix

$$C \in \mathbb{R}^{n\times n},$$

we have

$$\det C = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, c_{1,\pi(1)} c_{2,\pi(2)} \cdots c_{n,\pi(n)}.$$

Here S_n denotes the set of bijections

$$\pi : [n] \to [n].$$

Also, $\operatorname{sgn}(\pi)$ is the sign of π, i.e., -1 or 1 depending on the parity of

$$|\{(i,j) : i < j,\ \pi(i) > \pi(j)\}|.$$

If this number is even, then $\operatorname{sgn}(\pi) = 1$, and if it is odd, then $\operatorname{sgn}(\pi) = -1$.

Example 2.2. *Let* $\pi = 3241$. *Then* $\operatorname{sgn}(\pi) = 1$. *Its permutation matrix is*

$$P_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2.1 Proof of the Cauchy–Binet formula

Proof. Consider

$$A = \begin{array}{c} \\ m \\ n \end{array}\left[\begin{array}{c|c} m & n \\ \hline I_m & Q \\ \hline 0 & I_n \end{array}\right] \qquad B = \begin{array}{c} \\ m \\ n \end{array}\left[\begin{array}{c|c} n & m \\ \hline Q & 0 \\ \hline -I_n & R \end{array}\right]$$

and let

$$C = AB = \begin{array}{c} \\ m \\ n \end{array}\left[\begin{array}{c|c} n & m \\ \hline 0 & QR \\ \hline -I_n & R \end{array}\right].$$

Step 1: Compute $\det A$**.** Note that $\det A = 1$ since A is triangular and its diagonal entries are all 1's.

Step 2: Compute $\det C$**.** To find $\det C$, we need to pick permutations π of $[n+m]$. Lots of these terms will involve zero entries, in which case they do not add to the sum. So we need to focus on the π's that avoid zero entries.

So, π needs to assign each of the first n columns to its corresponding -1 entry in the $-I_n$ block of C. Then the last n rows of C are already assigned bijectively to the first n columns, so they cannot be assigned to a column among the last m; that is, they cannot be assigned to the R block of C.

Meanwhile, the first m rows can be assigned bijectively to any of the last m columns, i.e., to the QR block of C.

But then this shows that,

$$\det C = \pm \det(QR).$$

A more careful tracking of the -1's shows that

$$\det C = \det(QR).$$

Step 3: Compute $\det B$**.** To find $\det B$, we use a similar argument.

We need to consider permutations π avoiding zero summands.

The choice of m out of n columns for the first m rows fixes a set

$$K \in \binom{[n]}{m}$$

of columns.

To avoid 0's in the last n rows of B, for each column in $[n] \setminus K$, we have to pick its corresponding -1 in the $-I$ block of B.

But then we still need to pick columns for the rows K among the last n.

So, we are picking a permutation for $Q_{[m],K}$, and another permutation for $R_{K,[m]}$, i.e.,

$$\det\big(Q_{[m],K}\big) \cdot \det\big(R_{K,[m]}\big)$$

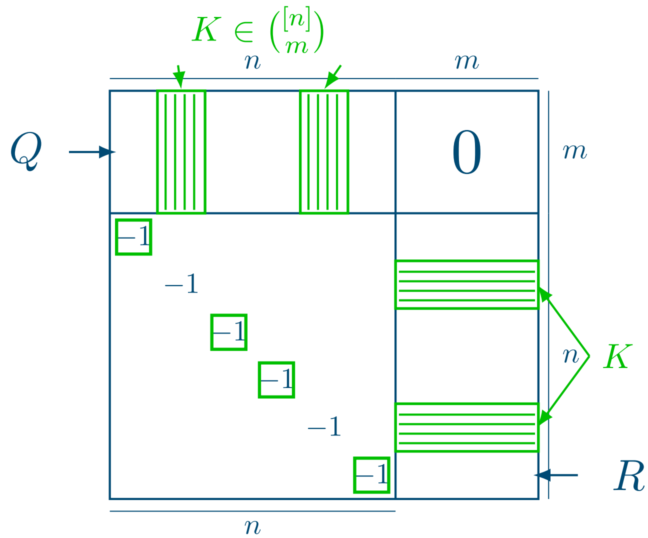

Figure 1: Matrix B

up to sign.

A more careful tracking of the -1's shows that the sign is correct.

Thus,

$$\det B = \sum_{K \in \binom{[n]}{m}} \det\bigl(Q_{[m],K}\bigr) \det\bigl(R_{K,[m]}\bigr).$$

Therefore,

$$\det(C) = \det(AB) = \det(A)\det(B) = \det(B)$$
$$\det(QR) = \sum_{K \in \binom{[n]}{m}} \det Q_{[m],K} \cdot \det R_{K,[m]}.$$

□

3 The Matrix–Tree Theorem

We now return to graphs.

Theorem 3.1 (Kirchhoff)**.** *Let G be an undirected graph and let $L = L(G)$ be its Laplacian matrix. Then, for any $i, j \in [n]$,*

$$|ST(G)| = \ell_{\hat{i},\hat{j}} = (-1)^{i+j} \det L_{\hat{i},\hat{j}},$$

where $\ell_{\hat{i},\hat{j}}$ denotes the (i,j)-cofactor of L.

Proof. We will only show the case in which $i = j = n$.

So,

$$\ell_{\hat{n},\hat{n}} = (-1)^{n+n} \det L_{\hat{n},\hat{n}} = \det L_{\hat{n},\hat{n}}.$$

Let D be any orientation of G, and let

$$B = B(D)$$

be the directed incidence matrix of D.

Last time we showed that

$$L = BB^T,$$

where L is independent of the choice of orientation D.

Let

$$W = [n-1].$$

Therefore,

$$L_{\hat{n},\hat{n}} = B_{W,E} \cdot (B_{W,E})^T.$$

By the Cauchy–Binet formula,

$$\det L_{\hat{n},\hat{n}} = \sum_{F \in \binom{E}{n-1}} \det(B_{W,F}) \cdot \det\big((B_{W,F})^T\big) = \sum_{F \in \binom{E}{n-1}} \big(\det(B_{W,F})\big)^2.$$

Note that $B_{W,F}$ has a combinatorial meaning. It is the directed incidence matrix of a digraph D_F with

$$V(D_F) = V, \qquad E(D_F) = F,$$

but for which we have removed the last row of $B(D_F)$, corresponding to node n.

We will say D_F is a tree if its corresponding underlying undirected graph is a tree.

If D_F is not a tree, then given that $|F| = n-1$, it must be disconnected. Thus, there is a component of D_F that does not contain node n.

The sum of the rows of $B_{W,F}$ corresponding to this component is zero (from our proposition last time). Hence,

$$\det(B_{W,F}) = 0.$$

If D_F is a tree, we can turn $B_{W,F}$ into a lower triangular matrix with all 1's along the diagonal so that

$$\det(B_{W,F}) = \pm 1.$$

We can do this inductively.

If $n = 1$, then $B_{W,F}$ is an empty matrix, with determinant 1 by convention.

If $n > 1$, then since D_F is a tree, it has at least two leaves, one of which must lie in $W = [n-1]$.

We can then do elementary row operations to relabel the graph such that the first row of $B_{W,F}$ corresponds to a leaf. Then $B_{W,F}$ has entry ± 1 in position $(1,1)$ and zeros everywhere else along this row given that this row corresponds to a leaf.

We then use induction. We have shown

$$\det(B_{W,F}) = \begin{cases} \pm 1, & \text{if } F \text{ forms a spanning tree of } G, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\sum_{F \in \binom{E}{n-1}} \big(\det(B_{W,F})\big)^2 = |ST(G)|.$$

Combining this with the earlier computation gives

$$\det L_{\hat{n},\hat{n}} = |ST(G)|.$$

Thus,

$$\ell_{\hat{n},\hat{n}} = |ST(G)|.$$

This proves the result in the case $i = j = n$. □

An Observation: Our proof shows that every square submatrix of B has determinant 0, 1, or -1. Hence B is totally unimodular. This is extremely useful in combinatorial optimization.