Lecture 24: The Spectrum of the Graph Laplacian

Date: April 22, 2026 **Scribe:** Simon Ruland

1 Graph Laplacian of K_n

Before moving on, consider K_n and $L = L(K_n)$. It consists of

1. $n-1$'s along the diagonal
2. -1's everywhere else

$$L(K_5) = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

So, $L_{\hat{n},\hat{n}}$ (removing the last column and the last row) is the same matrix except of shape $(n-1) \times (n-1)$.

Example 1.1.

$$L_{\hat{5},\hat{5}} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$$

Using elementary row operations, $L_{\hat{5},\hat{5}}$ *becomes*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

which is triangular and has determinant $\det(L_{\hat{5},\hat{5}}) = 5^3 = n^{(n-2)}$

More generally, this argument works for any n and we recover Cayley's formula.

$$|SP(K_n)| = \det(L_{\hat{n}\hat{n}})$$

2 Properties of the Graph Laplacian

Moving on, let $G = (V, E)$ be an undirected graph. Consider any orientation of E.

For $(i,j) \in E$, $(e_i - e_j)(e_i - e_j)^T$ is a matrix of zeros with $(i,i) = 1, (j,j) = 1, (i,j) = -1, (j,i) = -1$.

Much like building the graph one edge at a time, the graph Laplacian can be constructed in the following way:

$$L(G) = \sum_{(i,j) \in E} (e_i - e_j)(e_i - e_j)^T$$

Definition 2.1 (Positive semi-definite)**.** *$A \in \mathbb{R}^{n\times n}$ is positive semi-definite if*

$$x^T Ax \geq 0 \ \forall x \in \mathbb{R}^N$$

In this case, we write $A \succeq 0$.

For a symmetric $A \in \mathbb{R}^{(n\times n)}$, the following are equivalent

1. $A \succeq 0$
2. $A = VV^T$ for some V
3. A has non-negative eigenvalues

Remark 2.2. $L \succeq 0$

First note that L is symmetric.

First proof. If $A, B \succeq 0$, then
$A + B \succeq 0$ since $x^T(A + B)x = x^T Ax + x^T Bx \geq 0$ for all $x \in \mathbb{R}^n$
In particular, L is the sum of positive semi-definite matrices, so L itself is positive semi-definite. □

Second proof. Our first definition was $L = BB^T$, the directed incidence matrix □

Third proof. For any $x \in \mathbb{R}^n$

$$x^T Lx = x^T \left[\sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T \right] x$$
$$= \sum_{(i,j)\in E} x^T (e_i - e_j)(e_i - e_j)^T x$$

$= \sum_{(i,j)\in E}(x_i - x_j)^2$, which is always non-negative. □

Since L is positive semi-definite, we can write its eigenvalues as

$$0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$$

Note that e (the all 1's vector) is an eigenvector with the eigenvalue of 0.

$$Le = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T e = \sum_{(i,j)\in E} (e_i - e_j) \cdot 0 = 0$$

Thus, $\lambda_1 = 0$. This checks out since L is linearly dependent $\det(L) = 0$.
λ_2 is more interesting.

Theorem 2.3. *$\lambda_2 = 0$ iff G is disconnected.*

Proof. If G is disconnected, it can be partitioned as $G = G_1 \sqcup G_2$ with no shared edges.

Thus, after re-indexing, we can write $L = \begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix}$

Let e_{G_1} and e_{G_2} be the indicator vectors for the edges in G_1 and G_2 respectively.

$L_G e_{G_1} = L_G e_{G_2} = 0$

So e_{G_1} and e_{G_2} are eigenvectors of G with the eigenvalue of 0.

Also, $e_{G_1} e_{G_2}^T = 0$, so e_{G_1} and e_{G_2} are orthogonal. The eigen-space corresponding to $\lambda = 0$ has dimension at least 2.

$$\implies \lambda_2 = 0$$

Conversely, let x_2 be an eigenvector of L_G with $\lambda_2 = 0$.

We may assume $x_2^T e = 0$, possibly after finding an orthogonal basis fo the eigenspace of $\lambda = 0$, then

$$x_2^T L x_2 = x_2^T \lambda_2 x_2 = 0$$

Also, $x_2^T L x_2 = \sum_{(i,j) \in E} [x_2(i) - x_2(j)]^2 = 0$

$$\implies x_2(i) = x_2(j) \; \forall \; (i,j) \in E$$

Let $V_1 = \{i \in V x_2(i) \geq 0\}$ and $V_2 = \{i \in V x_2(i) < 0\}$

So, V_1 and V_2 share no edges.

But $x_2^T e = 0$ and $x_2 \neq 0$ (since it is an eigenvector), so x_2 must have both positive and negative entries. □

λ_2 is the "algebraic connectivity" of G.

More generally, one can compute this argument to show that $\lambda_k = 0$ iff G has at least k components.

We now present several lemmas without proof to demonstrate the utility of the spectrum of L.

Lemma 2.4. *$\lambda_2(G) \leq \kappa(G)$ where κ is the vertex connectivity of G.*

Lemma 2.5. *If $|V|$ is even, let $b(G)$ be the smallest bisection of G i.e.*

$$b(G) = \min_{S \subset V : |S| = |V - S|} |\delta(s)|$$

Then,

$$\frac{n}{4} \lambda_2(G) \leq b(G)$$

Lemma 2.6. *Let the maximum cut of G be $\max_{S \subseteq V} |\delta(s)|$, then*

$$\max_{S \subseteq V} |\delta(s)| \leq \lambda_n(L_G)$$

Lastly, we return to the matrix-tree theorem.

Theorem 2.7. *Let L_G have the eigenvalues*

$0 = \lambda_1 \leq \lambda_2 \leqq \ldots \leq \lambda_n$*, then*

$$|ST(G)| = \frac{1}{n} \prod_{i=2}^{n} \lambda_i$$

Before proving this, we need a fact from linear algebra:

$$\det(A+B)_{=} \sum_{S\subseteq[n]} \det(A_S)$$

where A_S is the matrix obtained from A after replacing each row a_i for $i \in S$ with row b_i of B. This follows from a multi-linear property of determinants.

Proof. If G is not connected, then $\lambda_2 = 0$ so

$$0 = |ST(G)| = \frac{1}{n}\lambda_2 \prod_{i=3}^{n} \lambda_i = 0$$

Suppose G is connected. Consider the characteristic polynomial of L_G

$$(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) = \lambda(\lambda - \lambda_2)...(\lambda - \lambda_n)$$

Thus, the term linear in λ is $(-1)^{n-1}\prod_{i=2}^{n}\lambda_i$
But also, the characteristic polynomial is defined as $\det(\lambda I - L)$.
Using the linear algebraic fact, the linear term in $\det(\lambda I - L)$ is

$$\sum_{S\subseteq[n]:|S|=n-1} \det(L_s) = (-1)^{n-1}\sum_{i=1}^{n} \det((L_G)_{\hat{i},\hat{i}}) = (-1)^{n-1} n|ST(G)|$$

□