Lecture 3: Combinatorial Arguments

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This lecture considers several strategies for constructing combinatorial proofs, including:

- "Counting two ways"
- (Strong) induction
- The "product principle" and the "sum principle"
- "Extremality" and sequences on finite structures

1 Counting Two Ways

The idea of this technique is to show that two expressions are equal by showing they both count the same set. As an example, consider the "handshaking lemma":

Theorem 1.1. *For any undirected graph $G = (V, E)$ we have that*

$$\sum_{u \in V} deg(u) = 2|E|.$$

Proof. Let $P = \{(u, e) \in V \times E : u \in e\}$. We first count this set as

$$|P| = \sum_{u \in V} |\{e \in E : u \in e\}| = \sum_{u \in V} \deg(u).$$

We now count this set as

$$|P| = \sum_{e \in E} |\{u \in V : u \in e\}| = 2|E|,$$

where the last equality holds since every edge contains exactly two nodes. □

2 Induction

The idea of an inductive proof is to first directly show that the statement we are trying to prove is true for a "small" case (e.g., $n = 0$). Then, we use what we assume (but soon will know) is true for "smaller" cases (e.g., for $1 < n \leq k$ for some k) to show that the statement is conditionally true for "larger" cases (e.g., $n = k+1$). The magic of induction is to transform our assumption to facts one step at a time given that we indeed started with a fact (e.g., which we directly showed for $n = 0$).

As an example of an inductive argument, we consider walks and paths on undirected graphs. A walk of length ℓ is a sequence $v_0, v_1, v_2, \ldots, v_\ell \in V$ such that

$$\{v_{i-1}, v_i\} \in E$$

for all $i \in [\ell]$. In other words, a walk is a generalization of paths that allows repetitions.

Theorem 2.1. *Let $G = (V, E)$ be an undirected graph. Then, any $\{s, t\}$-walk contains an $\{s, t\}$-path.*

Proof. Let W be an $\{s,t\}$-walk $s = v_0, v_1, \ldots, v_\ell = t$. For the base case, consider the case in which $\ell = 0$ Then, W is automatically a path.

By way of induction, suppose that $\ell \geq 1$ and that every $\{s,t\}$-walk of length $0 \leq k \leq \ell - 1$ contains an $\{s,t\}$-path. We need to show that every $\{s,t\}$-walk of length ℓ contains an $\{s,t\}$-path. If W is a walk but not a path, then it contains a repeated node $v_i = v_j$ such that $i < j$. Let $W' = v_0, v_1, \ldots, v_i, v_{j+1}, v_{j+2}, \ldots, v_\ell$. Then, W' is an $\{s,t\}$-walk of length strictly smaller than ℓ. By the inductive hypothesis, W' contains an $\{s,t\}$-path. Therefore, so does W. □

3 "Product Principle" and the "Sum Principle"

The idea of these proof techniques is that we can multiply the numbers of independent choices and sum over mutually exclusive and collectively exhaustive cases.

Theorem 3.1. *Let $V = [n]$. Then,*

1. *There are $2^{\binom{n}{2}}$ different undirected graphs on V.*

2. *There are $\binom{\binom{n}{2}}{k}$ different undirected graphs on V with $|E| = k$*

3. $\sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} = 2^{\binom{n}{2}}$

Proof. Since V is given, a graph $G = (V,E)$ is completely determined by our choice of E. We now prove each of the statements.

1. Note that $E \subseteq \{\{u,v\}\} : u \in V, v \in V\}$. That is, the set of candidate edges to construct our graph is of size $\binom{n}{2}$. Each edge can be included or excluded, so there are $2^{\binom{n}{2}}$ possible undirected graphs.

2. If $|E| = k$, then we must pick a set of size k from our set of candidates (without repetition). There are $\binom{\binom{n}{2}}{k}$ ways of doing this.

3. Any graph on G must have $0 \leq k \leq \binom{n}{2}$; these options are mutually exclusive and collectively exhaustive. Therefore, we use the sum principle to obtain

$$\sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} = 2^{\binom{n}{2}}$$

□

More generally, for any $m \in \mathbb{N}$, we have

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m$$

This expression sums over the rows of Pascal's Triangle (check out OEIS A007318).

4 "Extremality" and Finite Structures

Theorem 4.1. *Let $D = (V, A)$ be a directed graph. Then, $deg^-(u) = deg^+(u) = 1$ for all $u \in V$ if and only if D is the disjoint union of directed cycles.*

Proof. Per usual, one direction is "easy." If D is the disjoint union of directed cycles, then each $u \in V$ is in a unique directed cycle as its entire component. So, $\deg^-(u) = \deg^+(u) = 1$ for all $u \in V$.

To prove the other direction, consider any $v_1 \in V$. Since $\deg^+(v_1) = 1$, there is a unique $v_2 \in V$ such that $(v_1, v_2) \in A$. Similarly, there is a unique $v_3 \in V$ such that $(v_3, v_1) \in A$, and so on. Inductively, there is a sequence $v_1, v_2, v_3, \ldots$ with corresponding arcs. Since D is finite, at some point the sequence has a repetition. So, there exists a first (i.e., in this sense "extreme") pair $i < j$ with $v_i = v_j$. Note that we must have $v_i = v_1$ since otherwise $(v_{i-1}, v_i) \in A$ and also $(v_{j-1}, v_j) \in A$; and these arcs are distinct by our choice of "first" repetition. But then $\deg^-(v_i) \geq 2$, which contradicts the assumption that $\deg^-(u) = 1$ for all $u \in V$. So, $v_i = v_1$ and $v_1, v_2, ..., v_{j-1}, v_i = v_1$ forms a directed cycle. We can remove this cycle and further decompose the graph. □