Lecture 6: Permutations, sets, and multisets

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1 Permutations and factorials

You may have seen this before:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The first goal of today is to unpack this (for the case where $n \geq 0$).

$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ (n factorial)

n factorial is the number of different ways of ordering $[n]$
(number of permutations of $[n]$)

- You have n choices for which element of $[n]$ comes first.
- Given choice of first element, you have $n-1$ choices for which element comes second.
- Given choices of first and second, you have $n-2$ choices for third.
- And so on.

Now, we want to understand $\binom{n}{k}$, i.e., "n choose k" as the number of different ways of picking a subset of size k from a set of size n.

define:

$$\binom{[n]}{k} := \{T \subseteq [n] : |T| = k\}$$

what we mean with "n choose k" is:

$$\binom{n}{k} = \left|\binom{[n]}{k}\right|.$$

Let $P([n], k)$ be the set of words of length k using letters from $[n]$ without repetition.

- $|P([n], n)| = n!$

- $|P([n], k)| = \underbrace{n \times (n-1) \times \cdots \times (n-k+1)}_{k \text{ terms}}$

denoted $n_{\downarrow k} = \frac{n!}{(n-k)!}$

$\Rightarrow n_{\downarrow n} = n!$

Now, note that:

$$\left|\binom{[n]}{k}\right| k! = |P([n], k)|.$$

Since each set $T \in \binom{[n]}{k}$ can be ordered in $k!$ different ways.

$\Rightarrow \binom{n}{k} = \left|\binom{[n]}{k}\right| = \frac{|P([n],k)|}{k!} = \frac{n!}{(n-k)!k!}$

By convention:

- $\binom{n}{k} = 0$ if $k < 0$ or $k > n$
- $\binom{n}{0} = 1$
- $\binom{0}{0} = 1$

2 Some identities

Lemma 2.1. $\sum_k \binom{n}{k} = 2^n$

Proof. First, there are 2^n subsets of $[n]$. (recall product rule: $\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}$)

Also, a subset of $[n]$ must have some size k.
The sizes are mutually exclusive and collectively exhaustive.
Thus, use sum rule. □

Lemma 2.2. $\binom{n}{k} = \binom{n}{n-k}$

Proof. Let

$$f : \underbrace{2^{[n]}}_{\text{power set of } [n]} \to 2^{[n]},$$

where for $S \in \binom{[n]}{k}$, we let $f(S) = [n] \setminus S$.

Here, $[n] \setminus S$ denotes the set of all elements of $[n]$ that are **not in S**, and since S has k elements, $[n] \setminus S$ has $n-k$ elements.

Note:
$f \circ f = \mathrm{Id}_f \Rightarrow f$ is its own inverse — "an involution"

So f restricts to a bijection between $\binom{[n]}{k}$ and $\binom{[n]}{n-k}$.

$\binom{n}{k} = \left|\binom{[n]}{k}\right| = \left|\binom{[n]}{n-k}\right| = \binom{n}{n-k}$ □

Lemma 2.3. *For $n \geq 1$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.*

Proof. Let

$$\mathcal{S}_1 = \{S \in \binom{[n]}{k} : n \in S\}, \qquad \mathcal{S}_2 = \{S \in \binom{[n]}{k} : n \notin S\}.$$

Then

$$\binom{[n]}{k} = \mathcal{S}_1 \sqcup \mathcal{S}_2,$$

where $\sqcup$ denotes disjoint union.

By sum rule,

$$\binom{n}{k} = \left|\binom{[n]}{k}\right| = |\mathcal{S}_1| + |\mathcal{S}_2|.$$

If $S \in \mathcal{S}_1 \Rightarrow S - n \in \binom{[n-1]}{k-1}$.
$|\mathcal{S}_1| = \binom{n-1}{k-1}$

If $S \in \mathcal{S}_2 \xrightarrow{\text{bijection}} S \in \binom{[n-1]}{k}$.
$|\mathcal{S}_2| = \left|\binom{[n-1]}{k}\right| = \binom{n-1}{k}$

we plug these into $|\mathcal{S}_1| + |\mathcal{S}_2|$. □

3 Multisets

Multisets are unordered sets with repetitions accounted for.

For example,

$$\{a, a, b, b, c\} = \{a, b, a, b, c\} \text{ as a multiset.}$$

But

$$\{a, a, b, b, c\} \neq \{a, b, c\} \text{ as a multiset.}$$

Let $\left(\!\binom{[n]}{k}\!\right)$ as the set of multisets on $[n]$ of size k.

$$\left(\!\!\binom{n}{k}\!\!\right) = \left|\left(\!\!\binom{[n]}{k}\!\!\right)\right|.$$

Example 3.1. $\left(\!\binom{[3]}{2}\!\right) = \{\{1,1\},\{1,2\},\{1,3\},\{2,2\},\{2,3\},\{3,3\}\}$
$\left(\!\binom{3}{2}\!\right) = \left|\left(\!\binom{[3]}{2}\!\right)\right| = 6$

4 Dots and bars

Lemma 4.1. *For $n, k \geq 0$, $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$.*

Proof. We need to pick k things from n different kinds of things.

Form a list of "dots and bars".
We need k dots and $n-1$ bars.
This list will fully determine how many things of each kind we pick.

For example: $\left(\!\binom{3}{5}\!\right)$:
1) $\bullet\bullet \mid\mid \bullet\bullet\bullet \Rightarrow$ 2 of first kind, 0 of second kind, 3 of third kind.
2) $\mid \bullet\bullet\bullet \mid \bullet\bullet \Rightarrow$ 0 of first kind, 3 of second kind, 2 of third kind.

Our list of dots and bars is of length $k + n - 1 = n + k - 1$.

We know that k of the elements of the list must be dots (the rest are all bars).

How many different options for this choice? $\binom{n+k-1}{k}$. $\square$

5 Lattice paths

(Cartesian plane and its integer points)

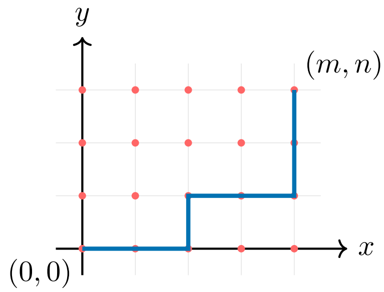

Figure 1: A lattice path from $(0, 0)$ to (m, n).

A lattice path from $(0, 0)$ to (m, n) is a path from $(0, 0)$ to (m, n) using only lattice moves allowed:

- North
- East

Lemma 5.1. *The number of N,E lattice paths from* $(0, 0)$ *to* (m, n) *is* $\binom{m+n}{m} = \binom{m+n}{n}$.

Proof. These paths are a list of size $m + n$ with m elements being E steps and n elements being N steps.

The choice of E steps fully determines the path.

There are $\binom{m+n}{m}$ choices. Identity is analogous. $\square$