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Lecture 8: Formal Power Series and Generating Functions
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1 Generating Polynomials

Definition 1.1 (Generating polynomial)**.** *A finite sequence* $a_0, a_1, a_2, \ldots, a_n$ *has the* generating polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i.$$

Example 1.2 (Binomial coefficients)**.** *Let* $a_k = \binom{n}{k}$ *for* $k = 0, 1, \ldots, n$ *(note: Pascal's triangle). The corresponding generating polynomial is*

$$f(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Theorem 1.3 (Binomial theorem)**.** *For* $n \in \mathbb{N}_0$*,*

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

In particular, this says that $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$ *has a "nice" factorization.*

Proof. We prove the identity by induction on n.
Base case. For $n = 0$,

$$\binom{0}{0} x^0 = 1 \cdot 1 = (1+x)^0 = 1.$$

Inductive step. We use the convention that $\binom{n}{k} = 0$ when $k < 0$ or $k > n$, so that

$$\sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=-\infty}^{\infty} \binom{n}{k} x^k.$$

Using the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we compute

$$\begin{aligned}
\sum_k \binom{n}{k} x^k &= \sum_k \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k \\
&= \sum_k \binom{n-1}{k-1} x^k + \sum_k \binom{n-1}{k} x^k \\
&= x \sum_k \binom{n-1}{k-1} x^{k-1} + \sum_k \binom{n-1}{k} x^k \\
&= x \sum_k \binom{n-1}{k} x^k + \sum_k \binom{n-1}{k} x^k \qquad \text{(re-indexing)} \\
&= x(1+x)^{n-1} + (1+x)^{n-1} \\
&= (1+x)^{n-1}(1+x) \\
&= (1+x)^n,
\end{aligned}$$

where we used the inductive hypothesis to replace $\sum_k \binom{n-1}{k} x^k$ by $(1+x)^{n-1}$. □

Combinatorial proof. Expanding

$$(1+x)^n = (1+x)(1+x)\cdots(1+x) \qquad (n \text{ times}),$$

the coefficient of x^k counts the number of ways to pick k of the factors from which we choose the x term (and from the remaining factors we choose 1). Hence there are $\binom{n}{k}$ ways, so the coefficient of x^k is $\binom{n}{k}$. □

Remark 1.4 (Some quick consequences)**.** *Evaluating at* $x = 1$ *gives*

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n.$$

Evaluating at $x = -1$ *gives*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0 \qquad \text{for } n \geq 1.$$

In particular,

$$\sum_{\substack{0 \leq k \leq n \\ k \text{ even}}} \binom{n}{k} = \sum_{\substack{0 \leq k \leq n \\ k \text{ odd}}} \binom{n}{k}.$$

2 Generating Functions

Definition 2.1 (Generating function)**.** *An infinite countable sequence* $a_0, a_1, a_2, a_3, \ldots$ *has the* generating function

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

We view $f(x)$ *as a* formal power series*: the powers of* x *are placeholders, and (unlike the generating polynomial case) we do not substitute numerical values for* x*.*

Example 2.2 (The all-ones sequence)**.** *Consider* $a_0 = 1, 1, 1, 1, \ldots$ *(all ones). Its generating function is*

$$f(x) = \sum_{k \geq 0} x^k = 1 + x + x^2 + x^3 + \cdots.$$

This has a "nice" representation as

$$f(x) = \frac{1}{1-x}.$$

Indeed, $f(x)$ *is the inverse of* $(1-x)$ *since*

$$f(x)(1-x) = (1 + x + x^2 + x^3 + \cdots)(1-x) = 1.$$

Definition 2.3 (Coefficient extraction)**.** *If* $A(x)$ *is a generating function, we denote*

$$a_n = [x^n]\, A(x),$$

to mean: extract the coefficient of x^n in $A(x)$.

Example 2.4.

$$[x^n]\left(\sum_{k\geq 0} x^k\right) = 1 \qquad \textit{for all } n \geq 0.$$

Remark 2.5. *Sometimes, a precise closed-form formula for*

$$a_n = [x^n]\, A(x)$$

can be easily extracted from the form of $A(x)$.

Remark 2.6 (Taylor series and coefficient extraction)**.** *In general, we can extract $[x^n]\, A(x)$ using a Taylor series expansion of $A(x)$ around $x = 0$:*

$$a_n = \frac{A^{(n)}(0)}{n!}.$$

This gives the numerical value of a_n (which can be obtained computationally).

3 The Algebra of Formal Power Series

Before going any further, we make the formal setting explicit.

Definition 3.1 (Formal power series over $\mathbb{C}$)**.**

$$\mathbb{C}[[x]] = \left\{ \sum_{n\geq 0} a_n x^n : a_n \in \mathbb{C} \textit{ for all } n \geq 0 \right\}.$$

We refer to $\mathbb{C}[[x]]$ as the "algebra of formal power series." (Note: $A(x) \in \mathbb{C}[[x]]$.)

Remark 3.2 (Ring operations)**.** *Let*

$$A(x) = \sum_{n\geq 0} a_n x^n, \qquad B(x) = \sum_{n\geq 0} b_n x^n.$$

- ***Addition:***

$$A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n) x^n.$$

- ***Scalar multiplication:*** *for $c \in \mathbb{C}$,*

$$cA(x) = c\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0} (c\, a_n) x^n.$$

- ***Multiplication (convolution):***

$$A(x) \cdot B(x) = \left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} c_n x^n =: C(x),$$

  where

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k}.$$

Remark 3.3 (Inverses and a word of caution)**.** *Recall that an inverse means* $S(x)S^{-1}(x) = 1$ *(by definition of the convolution product).*

Some statements that are true analytically *about a power series (in the sense seen in calculus) may not be true as statements in* $\mathbb{C}[[x]]$*. For example,* $\frac{1}{x}$ *makes sense analytically as the inverse of* x *(since* $x \cdot \frac{1}{x} = 1$*), but it does not make sense in* $\mathbb{C}[[x]]$ *since the formal power series*

$$x = 0 + x + 0x^2 + 0x^3 + \cdots$$

has no inverse in $\mathbb{C}[[x]]$*.*

Indeed, if $xf(x) = 1$*, then* $xf(x)$ *has* 0 *as its constant term, but the right-hand side of* $xf(x) = 1$ *is* 1*.*

Fact 3.4. *One can show that* $f(x) \in \mathbb{C}[[x]]$ *has an inverse if and only if*

$$\left[x^0\right] f(x) \neq 0.$$