Lecture 9: Solving Recurrence Relations
Date: February 18, 2026 **Scribe:** Noura Allugmani

1 Algorithm

Given $a_0, a_1, a_2 \ldots$ defined by a recurrence relation with some boundary conditions:

1. Multiply the r.r by x^n on both sides, and some over all $n \geqslant d$ where d is the smallest index for which r.r. is valid.
2. Define $A(x) = \sum_{n>0} a_n x^n$, and replace it in (1).
3. Solve for $A(x)$.
4. Find $a_n = [x^n]\, A(x)$ (ideally in closed-form by partial fraction expansion, and if not computationally).

Example 1.1. *$a_0 = 2$, and $a_n = 3a_{n-1}$ for $n \geq 1$. we know from visual inspection that*

$$a_n = 2 \cdot 3^n, \quad n \geqslant 0.$$

Goal: Recover this using the algorithm.

1.

$$\Rightarrow a_n x^n = 3a_{n-1}x^n$$

$$\Rightarrow \sum_{n\geqslant 1} a_n x^n = \sum_{n \geqslant 1} 3a_{n-1}x^n$$

2.

$$A(x) = \sum_{n \geq 0} a_n x^n$$

Note:

- $$\sum_{n\geqslant 1} a_n x^n = A(x) - a_0 = A(x) - 2$$
- $$\begin{aligned} \sum_{n\geqslant 1} 3a_{n-1}x^n &= 3x \sum_{n\geqslant 1} a_{n-1}x^{n-1} \\ &= 3x \sum_{n\geqslant 0} a_n x^n \\ &= 3xA(x) \end{aligned}$$

$\Rightarrow A(x) - 2 = 3xA(x).$

3.

$$\begin{aligned}
A(x) - 3xA(x) &= 2\\
A(x)(1-3x) &= 2\\
A(x) &= \frac{2}{1-3x}.
\end{aligned}$$

4. Extract $a_n = [x^n]\,A(x)$ *by expanding* $\dfrac{2}{1-3x}$ *as a "geometric series". We can substitute* $3x$ *for* x *in* $\dfrac{1}{1-x}$ *to obtain* $A(x) = \sum_{n\geqslant 0} 2\cdot 3^n x^n$. $1-3x$ *is shorthand for* $1-3x+0\cdot x^2+0\cdot x^3+\cdots$, *and* $\dfrac{1}{1-3x} = 1+3x+3^2x^2+\cdots$ *is the inverse of* $1-3x+0\cdot x^2+0\cdot x^3+\ldots$. *You should convince yourself that* $\left(1+3x+3^2x^2+\cdots\right)\left(1-3x+0x^2+0x^3+\cdots\right)=1$, *where 1 is shorthand for* $1+0\cdot x+0\cdot x^2+\cdots$

$$\begin{aligned}
c_0 &= 1\cdot 1 = 1 \checkmark\\
c_1 &= 1(-3)+(3)(1) = 0 \checkmark\\
c_n &= (1\cdot 0)+(3)(-3)+3^2(1) = 0 \checkmark
\end{aligned}$$

Example 1.2. $a_0 = 1, a_1 = -4$, *and* $a_n = 4a_{n-1} - 4a_{n-2}, \forall n \geq 2$.

1.

$$\begin{aligned}
&\Rightarrow a_n x^n = \left(4a_{n-1} - 4a_{n-2}\right)x^n\\
&\Rightarrow \sum_{n\geqslant 2} a_n x^n = \sum_{n\geqslant 2}\left(4a_{n-1} - 4a_{n-2}\right)x^n.
\end{aligned}$$

2.

$$A(x) = \sum_{n\geqslant 0} a_n x^n$$

LHS.

$$\begin{aligned}
\sum_{n\geqslant 2} a_n x^n &= A(x) - a_0 - a_1 x.\\
&= A(x) - 1 + 4x.
\end{aligned}$$

RHS.

$$\begin{aligned}
&\sum_{n\geqslant 2}\left(4a_{n-1} - 4a_{n-2}\right)x^n\\
=&4\sum_{n\geqslant 2} a_{n-1}x^n - 4\sum_{n\geqslant 2} a_{n-2}x^n\\
=&4x\sum_{n\geqslant 2} a_{n-1}x^{n-1} - 4x^2\sum_{n\geqslant 2} a_{n-2}x^{n-2}\\
=&4x\sum_{n\geqslant 1} a_n x^n - 4x^2\sum_{n\geqslant 0} a_n x^n\\
=&4x\left(A(x) - a_0\right) - 4x^2 A(x) = 4x\left(A(x) - 1\right) - 4x^2 A(x)\\
\Rightarrow& A(x) - 1 + 4x = 4x(A(x) - 1) - 4x^2 A(x).
\end{aligned}$$

3. $A(x) = \dfrac{1-8x}{1-4x+4x^2}$

4. Note that:

$A(x) = \dfrac{4}{1-2x} - \dfrac{3}{(1-2x)^2}$ *by partial fraction expansion.* $\dfrac{1}{1-2x}$ *looks like something we already know how to deal with.* $\dfrac{1}{1-x} = \sum_{n\geqslant 0} x^n$*, by replacing* $2x$ *for* x *we have*

$$\frac{1}{1-2x} = \sum_{n\geqslant 0} 2^n x^n$$
$$\Rightarrow \frac{4}{1-2x} = \sum_{n\geqslant 0} 4\cdot 2^n x^n$$

we will not prove this, but it is a fact that

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \left(\!\!\binom{k}{n}\!\!\right) x^n.$$

$\Rightarrow$ *we follow same strategy of replacing* $2x$ *for* x *in expansion above to find*

$$\frac{3}{(1-2x)^2} = \sum_{n\geqslant 0} 3\binom{n+1}{1} 2^n x^n.$$
$$= A(x) = \sum_{n\geqslant 0}\left(4\cdot 2^n - 3\binom{n+1}{1}2^n\right) x^n.$$

Without proof: the ability of getting a closed form solution as a linear combination of the reciprocals of the roots of the denominator of the corresponding generating function is possible if and only if your recurrence relation is linear of fixed number of terms d. (This is to have a rational generating function.)

$$a_{n+d} = c_1 a_{n+d-1} + c_2 a_{n+d-2} + \ldots + c_d a_n \text{ with } c_d \neq 0.$$

Example 1.3. *The Fibonacci numbers.*

$$f_0 = 0, f_1 = 1, \text{ and } f_n = f_{n-1} + f_{n-2}, n \geqslant 2.$$

1.

$$\Rightarrow f_n x^n = (f_{n-1} + f_{n-2})\, x^n.$$
$$\sum_{n\geqslant 2} f_n x^n = \sum_{n\geqslant 2} (f_{n-1} + f_{n-2})\, x^n.$$

2.

$$F(x) = \sum_{n\geqslant 0} f_n x^n$$

LHS.

$$\sum_{n\geq 2} f_n x^n = F(x) - f_0 - f_1 x$$
$$= F(x) - x$$

RHS.

$$\begin{aligned}
\sum_{n\geqslant 2}\left(f_{n-1}+f_{n-2}\right)x^n &= \sum_{n\geqslant 2} f_{n-1}x^n + \sum_{n\geqslant 2} f_{n-2}x^n \\
&= x\sum_{n\geqslant 2} f_{n-1}x^{n-1} + x^2\sum_{n\geqslant 2} f_{n-2}x^{n-2} \\
&= x\sum_{n\geqslant 1} f_n x^n + x^2\sum_{n\geqslant 0} f_n x^n \\
&= x\left(F(x)-f_0\right) + x^2F(x) \\
&= xF(x) + x^2F(x). \\
\Rightarrow F(x) - x &= xF(x) + x^2F(x)
\end{aligned}$$

3. *Solve for $F(x)$ to find $F(x) = \dfrac{x}{1-x-x^2}$.*

4. *Find $f_n = [x^n]\,F(x)$ as*

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We will find f_n next time, but for now be amazed that it turns out to be always an integer.