Problem Set 2 **Due Date:** March 13, 2026

1. The Lucas numbers are given by $\ell_0 = 2$, $\ell_1 = 1$, and $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \geq 2$. Recall that the Fibonacci numbers are given by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, and that f_n is the number of matchings in the $(n-1)$-path graph P_{n-1} for $n \geq 1$.
   (a) Show that $\ell_n = f_{n+1} + f_{n-1}$ for $n \geq 1$.
   (b) Show that ℓ_n is the number of matchings in the n-cycle graph C_n for $n \geq 3$.
   (c) Show that $\ell_{m+n} = f_{m-1}\ell_n + f_m \ell_{n+1}$ for $n, m \geq 1$ through an inductive argument.
   (d) Show that $f_{2n} = f_n \ell_n$ for $n \geq 1$ through a combinatorial argument.
2. Recall that the nth Catalan number c_n, which is given by $c_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=1}^{n} c_{i-1}c_{n-i}$ with $c_0 = 1$, counts the number of Dyck paths of semilength $n \geq 1$. Show that the following combinatorial objects are also counted by c_n:
   (a) The weakly increasing sequences $a_1 \leq a_2 \leq \cdots \leq a_n$ such that $a_i \leq i$ for all $i \in [n]$.
   (b) The noncrossing partitions $\rho = B_1 | \ldots | B_k \vdash [n]$. A crossing is any $a < b < c < d$ such that $a, c \in B_i$, $b, d \in B_j$, and $i \neq j$.

   You may use any argument or combination of arguments: closed-form formulas, recursive relations, bijections, and/or generating functions.
3. Derive the generating function and closed-form formula for the Lucas numbers.
4. Prove that, for the n-cycle graph C_n, $P(C_n; t) = (t-1)^n + (-1)^n(t-1)$.