Problem Set 3 **Due Date:** April 13, 2026

1. Let $G = (V, E)$ be a directed graph with source $s \in V$, target $t \in V$, and capacities $c : E \to \mathbb{R}_{\geq 0}$. Give a flow-based algorithm to decide whether G has a unique minimum s-t cut. Show its correctness.

2. Let $G = (V, E)$ be an undirected graph and $d^* \in \mathbb{R}_{>0}$. The degree density of $\emptyset \neq S \subseteq V$ is given by
$$\rho(S) = \frac{|E[S]|}{|S|},$$
where $G[S] = (S, E[S])$ is the subgraph of G induced by S. Give a flow-based algorithm to decide whether there exists $\emptyset \neq S \subseteq V$ such that $\rho(S) > d^*$. Show its correctness.

3. Let $[n+1] \coloneqq \{1, 2, \ldots, n, n+1\}$ be the set of teams in a baseball league. At the time of writing, each team $i \in [n+1]$ has won $w_i \in \mathbb{N}_0$ games. Moreover, each pair of teams $i, j \in [n+1]$ with $i < j$ has $g_{ij} \in \mathbb{N}_0$ games left to play against one another until the end of the season: there are no ties in baseball, so on each of these games, exactly one of i or j will win. Your local team $n+1$, also known as the "Colorado Pebbles," has not been doing so well this season. However, you are one hopeful fan: you plan to go to every home game until it is a mathematical impossibility for the Colorado Pebbles to come out first in the final season ranking (possibly as part of a tie). Give a flow-based algorithm to decide whether the Colorado Pebbles have been eliminated as contenders for the first place in the final season ranking at the time of writing. Show its correctness. To determine mathematical (im)possibility, you may assume that the Colorado Pebbles win all their remaining games.